Transformation Groups, Vol. ?, No. ?, ??, pp. 1–?? ©Birkhäuser Boston (??)

RESTRICTED LIE ALGEBRAS WITH MAXIMAL 0-PIM

JÖRG FELDVOSS

Department of Mathematics and Statistics
University of South Alabama
Mobile, AL 36688-0002, USA
jfeldvoss@southalabama.edu

SALVATORE SICILIANO

Dipartimento di Matematica e Fisica "Ennio De Giorgi"
Università del Salento
Via Provinciale Lecce-Arnesano
I-73100 Lecce, Italy
salvatore.siciliano@unisalento.it

THOMAS WEIGEL

Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano-Bicocca
Via Roberto Cozzi 55
I-20125 Milano, Italy
thomas.weigel@unimib.it

Dedicated to Otto H. Kegel on the occasion of his eighty-first birthday

Abstract. In this paper it is shown that the projective cover of the trivial irreducible module of a finite-dimensional solvable restricted Lie algebra is induced from the one-dimensional trivial module of a maximal torus. As a consequence, the number of the isomorphism classes of irreducible modules with a fixed p-character for a finite-dimensional solvable restricted Lie algebra L is bounded above by $p^{\mathrm{MT}(L)}$, where $\mathrm{MT}(L)$ denotes the maximal dimension of a torus in L. Finally, it is proved that in characteristic $p > 3$ the projective cover of the trivial irreducible L-module is induced from the one-dimensional trivial module of a torus of maximal dimension, only if L is solvable.

Introduction

For a fixed prime number p Wolfgang Willems considered the class $\mathcal{P}_0(p)$ of all finite groups G for which the dimension of the projective cover of the one-dimensional trivial $\mathbb{F}G$-module over a field $\mathbb{F}$ of characteristic p has minimal dimension and compared $\mathcal{P}_0(p)$ to the class of all finite groups having a p-complement. In particular, every p-solvable group belongs to $\mathcal{P}_0(p)$, but the converse is not true (see [33, Section 4]). More recently, Gunter Malle and the third author of this

Received October 12, 2014. Accepted October 9, 2015.

paper classified all finite non-abelian simple groups belonging to $\mathcal{P}_0(p)$ for a fixed prime number p by using the classification of finite simple groups (see [17, Theorem A]). As a consequence, they obtained that a finite group G is solvable (i.e., G is p-solvable for every prime number p) if, and only if, $G \in \mathcal{P}_0(p)$ for every prime number p (see [17, Corollary B]).

In this paper we investigate an analogous question for finite-dimensional restricted Lie algebras over a field of prime characteristic. It turns out that for a restricted Lie algebra there is a canonical upper bound for the dimension of the projective cover of its one-dimensional trivial module (see Proposition 1.2). We say that a finite-dimensional restricted Lie algebra has *maximal* 0*-PIM* if this maximal possible dimension is attained (see Section 2). One main goal of the paper is then to classify all finite-dimensional restricted Lie algebras having maximal 0-PIM. We prove that a finite-dimensional restricted Lie algebra over a field of characteristic $p > 3$ has maximal 0-PIM if, and only if, it is solvable (see Theorem 6.4). A main ingredient of the proof is the classification of finite-dimensional simple Lie algebras of absolute toral rank two over an algebraically closed field of characteristic $p > 3$ due to Sasha Premet and Helmut Strade (see [19], [20], and [21]). A comprehensive exposition of the classification of finite-dimensional simple Lie algebras of arbitrary absolute toral rank can be found in the volumes of Helmut Strade (see [26], [28], [29]) and also in the survey [22]. Although not being sufficient for our purposes it should be mentioned that the classification of the restricted simple Lie algebras over an algebraically closed field of characteristic $p > 7$ by Richard Block and Robert Wilson (see [1]) was an essential step for the classification result of Sasha Premet and Helmut Strade.

We show by an example that in characteristic 2 there exists a finite-dimensional non-solvable restricted Lie algebra having maximal 0-PIM. In characteristic 3 we do not know of such a counterexample, but according to the lack of a classification of finite-dimensional simple Lie algebras of absolute toral rank two in this case, at the moment it is not clear whether our result holds in characteristic 3.

In the first section we collect several useful results for projective covers of modules over reduced enveloping algebras that will be needed later in the paper. Some of these results were already known in special cases, but for the convenience of the reader we treat them here in one place. The most important one is Theorem 1.5, which is the Lie-theoretic analogue of a result for finite-dimensional group algebras due to Wolfgang Willems (see [33, Lemma 2.6] or also [15, Lemma VII.14.2]). It will be used in the proof of the main result of the next section, namely, that every finite-dimensional solvable restricted Lie algebra has maximal 0-PIM (see Theorem 2.2). In Section 3 this result is employed to establish an upper bound for the number of the isomorphism classes of irreducible modules with a fixed p-character for a solvable restricted Lie algebra (see Theorem 3.1). This generalizes the known results for nilpotent (see [24, Satz 6]) and supersolvable restricted Lie algebras (see [11, Theorem 4]). For quite some time the first author has conjectured that this bound holds for *every* finite-dimensional restricted Lie algebra (see also [14, Section 10, Conjecture] for simple Lie algebras of classical type). In the fourth section we prove a general result on the compatibility of induction for any filtered restricted Lie algebra of finite depth and finite height with the associated graded restricted Lie algebra which might be of general interest. In particular, the irreducibility of the induced module for the associated graded restricted Lie algebra implies the irreducibility of the corresponding induced module for the filtered restricted Lie algebra (see Theorem 4.3). Section 5 discusses some non-graded Hamiltonian Lie algebras and their representations. Here the irreducibility of certain induced modules is obtained from Theorem 4.3 and the known corresponding result for the associated graded restricted Lie algebra (see Theorem 5.3). The last section is devoted to a proof of the converse of Theorem 2.2 in characteristic $p > 3$ and a counterexample to this result in characteristic 2. In characteristic 3 this converse seems to be open, and we hope to come back to this on another occasion.

In the following we briefly recall some of the notation that will be used in this paper. Let $\langle X\rangle_{\mathbb{F}}$ denote the $\mathbb{F}$-subspace of a vector space V over a field $\mathbb{F}$ spanned by a subset X of V. For a subset X of a restricted Lie algebra L we denote by $\langle X\rangle_p$ the p-subalgebra of L generated by X. Finally, $[L, L]$ or $L^{(1)}$ will denote the derived subalgebra of a Lie algebra L, and $L^{(2)}$ will denote the derived subalgebra of $L^{(1)}$. For more notation and some well-known results from the structure and representation theory of modular Lie algebras we refer the interested reader to Chapters 1 – 5 in [30] as well as to Chapters 1, 4, and 7 in [26].

1. Projective covers of modules with a p-character

Let A be a finite-dimensional unital associative algebra, and let M be a (unital left) A-module. Recall that a projective module $P_A(M)$ is a *projective cover* of M, if there exists an A-module epimorphism π_M from $P_A(M)$ onto M such that the kernel of π_M is contained in the radical of $P_A(M)$. If projective covers exist, then they are unique up to isomorphism (see also the remark after (PC) below). It is well known that projective covers of finite-dimensional modules over finite-dimensional associative algebras always exist and are again finite-dimensional. Moreover, every projective indecomposable A-module is isomorphic to the projective cover of its irreducible head. In this way one obtains a bijection between the isomorphism classes of the projective indecomposable A-modules and the isomorphism classes of the irreducible A-modules. The following universal property of the pair $(P_A(M), \pi_M)$, which is a direct consequence of Nakayama's lemma, is well known.

(PC) If P is a projective A-module and π is an A-module epimorphism from P onto M, then every A-module homomorphism η from P into $P_A(M)$ with $\pi_M \circ \eta = \pi$ is an epimorphism.

Remark. Since P is projective and π_M is an epimorphism, there always exists an A-module homomorphism η from P to $P_A(M)$ such that $\pi_M \circ \eta = \pi$. In particular, it follows from (PC) that projective covers are unique up to isomorphism.

Let L be a finite-dimensional restricted Lie algebra over a field of prime characteristic p, and let χ be a linear form on L. Moreover, let $u(L, \chi)$ denote the χ-reduced universal enveloping algebra of L (see [31, §1.3] or [30, p. 212]), and let $P_L(M) := P_{u(L,\chi)}(M)$ denote the projective cover of the $u(L, \chi)$-module M. If H is a p-subalgebra of L, and V is an H-module with p-character $\chi_{|H}$, then we set $\mathrm{Ind}_H^L(V, \chi) := u(L, \chi) \otimes_{u(H,\chi_{|H})} V$ (see [31, §1.3] or [30, Section 5.6] for the usual properties of induction). The following fact, which will be used in the proof of Theorem 6.4, is included here for completeness.

Proposition 1.1. *Let L be a finite-dimensional restricted Lie algebra over a field of prime characteristic p, let χ be a linear form on L, let H be a p-subalgebra of L, and let M be a finite-dimensional L-module with p-character χ. Then $P_L(M)$ is a direct summand of $\operatorname{Ind}_H^L(P_H(M_{|H}),\chi)$.*

Proof. Set $P := \operatorname{Ind}_H^L(P_H(M_{|H}),\chi)$. Since induction preserves projectivity, P is a projective $u(L,\chi)$-module. As $P_H(M_{|H})$ is the projective cover of $M_{|H}$, there is an H-module epimorphism π from $P_H(M_{|H})$ onto $M_{|H}$. According to the universal property of induced modules (see [30, Theorem 5.6.3]), there exists an L-module epimorphism from P onto M. Then (PC) and the subsequent remark imply that $P_L(M)$ is a homomorphic image of P, and thus also a direct summand of P. □

Let

$$\operatorname{MT}(L) := \max\{\dim_{\mathbb{F}} T \mid T \text{ is a torus of } L\}$$

denote the *maximal dimension* of a torus in L (see [26, Notation 1.2.5]). The next result is a consequence of Proposition 1.1 and the semisimplicity of reduced universal enveloping algebras of tori.

Proposition 1.2. *Let L be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic p, let χ be a linear form on L, and let M be a finite-dimensional L-module with p-character χ. Then $P_L(M)$ is a direct summand of $\operatorname{Ind}_T^L(M_{|T},\chi)$ for any torus T of L. In particular,*

$$\dim_{\mathbb{F}} P_L(M) \leq (\dim_{\mathbb{F}} M)\cdot p^{\dim_{\mathbb{F}} L-\operatorname{MT}(L)}\,.$$

Proof. Let T be any torus of L. By virtue of the semisimplicity of $u(T,\chi_{|T})$ (see [9, Lemma 3.2]), $M_{|T}$ is a projective $u(T,\chi_{|T})$-module, and it follows from Proposition 1.1 that $P_L(M)$ is a direct summand of $\operatorname{Ind}_T^L(M_{|T},\chi)$.

Assume now in addition that T is of maximal dimension in L, i.e., one has $\dim_{\mathbb{F}} T = \operatorname{MT}(L)$. Then it follows from the first part and the formula for the dimension of induced modules (see [30, Proposition 5.6.2]) that

$$\dim_{\mathbb{F}} P_L(M) \leq \dim_{\mathbb{F}} \operatorname{Ind}_T^L(M_{|T},\chi) = (\dim_{\mathbb{F}} M)\cdot p^{\dim_{\mathbb{F}} L-\operatorname{MT}(L)}\,.$$

This yields the claim. □

The following property is a consequence of the first part of Proposition 1.2 and the formula for the dimension of induced modules (see [30, Proposition 5.6.2]):

Corollary 1.3. *Let L be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic p, and let χ be a linear form on L. If M is a finite-dimensional L-module with p-character χ such that*

$$\dim_{\mathbb{F}} P_L(M) = (\dim_{\mathbb{F}} M)\cdot p^{\dim_{\mathbb{F}} L-\operatorname{MT}(L)}\,,$$

then $P_L(M) \cong \operatorname{Ind}_T^L(M_{|T},\chi)$ holds for any torus T of L of maximal dimension.

The proof of the following result, which will be used in Section 3, follows the line of the proof of [8, Theorem 2]. Here the trivial irreducible irreducible L-module of a restricted Lie algebra L over a field $\mathbb{F}$ will be denoted also by $\mathbb{F}$.

Proposition 1.4. *Let L be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic p, let χ be a linear form on L, and let M be a non-zero finite-dimensional L-module with p-character χ. Then $P_L(\mathbb{F})$ is a direct summand of $P_L(M) \otimes M^*$.*

Proof. Set $P := P_L(M) \otimes M^*$. According to [30, Theorem 5.2.7(2)] and [7, Lemma 2.3], P is a projective $u(L, 0)$-module. As $P_L(M)$ is the projective cover of M, there is an L-module epimorphism π_M from $P_L(M)$ onto M. Tensoring π_M with the identity transformation of M^* yields an L-module epimorphism from P onto $M \otimes M^*$ which in turn has $\mathbb{F}$ as an epimorphic image. Now one may proceed as in the proof of Proposition 1.1. □

The following dimension formula is the Lie-theoretic analogue of a result for finite-dimensional group algebras due to Wolfgang Willems (see [33, Lemma 2.6] or [15, Lemma VII.14.2]). It will be important for the proof of the first main result of this paper, which can be found in the next section.

Theorem 1.5. *Let L be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic p, let χ be a linear form on L, and let I be a p-ideal of L. Then*

$$\dim_{\mathbb{F}} P_L(S) = \dim_{\mathbb{F}} P_I(\mathbb{F}) \cdot \dim_{\mathbb{F}} P_{L/I}(S)$$

holds for every irreducible L-module S with p-character χ such that $I \cdot S = 0$.

Proof. As both the restriction functor from L to I and the induction functor from I to L are exact additive covariant functors that map projectives to projectives, their composition $\mathcal{F} := \operatorname{Ind}_I^L(-, \chi)_{|I}$ has the same properties. It follows from the generalized Cartan-Weyl identity (see [30, Lemma 5.7.1]) that I annihilates the module $\mathcal{F}(\mathbb{F})$ which is therefore isomorphic to $\dim_{\mathbb{F}} u(L/I, \chi)$ copies of the trivial irreducible I-module. The exactness of $\mathcal{F}$ and (PC) implies that $\mathcal{F}(P_I(\mathbb{F}))$ contains a direct summand that is isomorphic to $\dim_{\mathbb{F}} u(L/I, \chi)$ copies of $P_I(\mathbb{F})$. Since both modules have the same dimension, they are isomorphic.

As S is a trivial I-module, there exists a non-zero I-module homomorphism from $P_I(\mathbb{F})$ to $S_{|I}$, and one obtains from the universal property of induced modules (see [30, Theorem 5.6.3]) and the irreducibility of S that there exists an L-module epimorphism from $\operatorname{Ind}_I^L(P_I(\mathbb{F}), \chi)$ onto S. But the former module is projective, and thus it follows from (PC) that $P_L(S)$ is an epimorphic image of $\operatorname{Ind}_I^L(P_I(\mathbb{F}), \chi)$. Hence $P_L(S)$ is a direct summand of $\operatorname{Ind}_I^L(P_I(\mathbb{F}), \chi)$. Then after restriction to I the isomorphism $\mathcal{F}(P_I(\mathbb{F})) \cong P_I(\mathbb{F})^{\oplus \dim_{\mathbb{F}} u(L/I,\chi)}$ in conjunction with the Krull-Remak-Schmidt Theorem implies that $P_L(S)_{|I} \cong P_I(\mathbb{F})^{\oplus e}$ for some positive integer e. In particular, one obtains that $\dim_{\mathbb{F}} P_L(S) = e \cdot \dim_{\mathbb{F}} P_I(\mathbb{F})$.

Let $\mathcal{D}$ denote the left adjoint functor of the inflation functor $\mathcal{I}$ from L/I to L. Then $\mathcal{D}$ is just the coinvariants functor so that $\mathcal{D}(M) = M/IM$ for every $u(L, \chi)$-module M. Since $\mathcal{I}$ is obviously exact, it follows that $\mathcal{D}$ maps projectives to projectives (see [32, Proposition 2.3.10]). For any irreducible $u(L/I, \chi)$-module M one has

$$\begin{aligned} \operatorname{Hom}_{L/I}(\mathcal{D}(P_L(S)), M) &\cong \operatorname{Hom}_L(P_L(S), \mathcal{I}(M)) \cong \operatorname{Hom}_L(S, \mathcal{I}(M)) \\ &\cong \operatorname{Hom}_{L/I}(S, M) \cong \operatorname{Hom}_{L/I}(P_{L/I}(S), M)\,. \end{aligned}$$

Hence (PC) implies that $\mathcal{D}(P_L(S))$ and $P_{L/I}(S)$ are isomorphic. As $P_I(\mathbb{F})/IP_I(\mathbb{F})$ is one-dimensional, one concludes from $P_L(S)_{|I} \cong P_I(\mathbb{F})^{\oplus e}$ that

$$e = \dim_{\mathbb{F}} P_L(S)/IP_L(S) = \dim_{\mathbb{F}} \mathcal{D}(P_L(S)) = \dim_{\mathbb{F}} P_{L/I}(S)\,.$$

Consequently, the assertion follows from the last equality of the previous paragraph. □

Remark. The above proof would also work in the case of finite-dimensional group algebras. In particular, this provides an alternative proof of Wolfgang Willems' result [33, Lemma 2.6].

2. The 0-PIM of a solvable restricted Lie algebra

Let L be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic. It follows from Proposition 1.2 that

$$\dim_{\mathbb{F}} P_L(\mathbb{F}) \leq p^{\dim_{\mathbb{F}} L - \mathrm{MT}(L)}\,. \tag{2.1}$$

We say that L has *maximal* 0*-PIM* if $\dim_{\mathbb{F}} P_L(\mathbb{F}) = p^{\dim_{\mathbb{F}} L - \mathrm{MT}(L)}$. In this case it follows from Corollary 1.3 that $P_L(\mathbb{F}) \cong \mathrm{Ind}_T^L(\mathbb{F}, 0)$ for any torus T in L of maximal dimension. Our goal in this section is to prove that any finite-dimensional solvable restricted Lie algebra L has maximal 0-PIM. In particular, the 0-PIM is induced from the one-dimensional trivial module of a torus of maximal dimension. The next result will be important in the induction step of the proof of Theorem 2.2 and in the proof of Theorem 6.4 (see also [15, Theorem VII.14.3] for the group-theoretic analogue).

Lemma 2.1. *Let L be a finite-dimensional restricted Lie algebra over a field of prime characteristic p, and let I be a p-ideal of L. Then L has maximal* 0*-PIM if, and only if, I and L/I have maximal* 0*-PIM.*

Proof. Suppose first that L has maximal 0-PIM. Recall that

$$\mathrm{MT}(L) = \mathrm{MT}(I) + \mathrm{MT}(L/I)$$

holds for any p-ideal I of L (see [26, Lemma 1.2.6(2)(a)]). Then Theorem 1.5 in conjunction with this formula yields

$$[\dim_{\mathbb{F}} P_I(\mathbb{F})] \cdot [\dim_{\mathbb{F}} P_{L/I}(\mathbb{F})] = p^{\dim_{\mathbb{F}} I - \mathrm{MT}(I)} \cdot p^{\dim_{\mathbb{F}} L/I - \mathrm{MT}(L/I)}\,. \tag{2.2}$$

Suppose that $\dim_{\mathbb{F}} P_I(\mathbb{F}) < p^{\dim_{\mathbb{F}} I - \mathrm{MT}(I)}$. Then $\dim_{\mathbb{F}} P_{L/I}(\mathbb{F}) > p^{\dim_{\mathbb{F}} L/I - \mathrm{MT}(L/I)}$, which contradicts (2.1). Hence $\dim_{\mathbb{F}} P_I(\mathbb{F}) = p^{\dim_{\mathbb{F}} I - \mathrm{MT}(I)}$, and therefore also $\dim_{\mathbb{F}} P_{L/I}(\mathbb{F}) = p^{\dim_{\mathbb{F}} L/I - \mathrm{MT}(L/I)}$. Consequently, I and L/I must have maximal 0-PIM. The other implication follows from (2.2) in a similar way. □

From Lemma 2.1 one may deduce the first main result of this paper.

Theorem 2.2. *Every finite-dimensional solvable restricted Lie algebra over a field of prime characteristic has maximal* 0*-PIM.*

Proof. We may proceed by induction on the dimension of the solvable restricted Lie algebra L. If L is abelian, then the assertion is a consequence of [6, Satz II.3.2] (see also [8, Corollary 1]). In particular, it holds if $\dim_{\mathbb{F}} L = 1$. Thus we may assume that L is not abelian, and that the claim holds for all Lie algebras of dimension less than $\dim_{\mathbb{F}} L$.

As L is solvable but not abelian, the p-subalgebra $I := \langle [L, L] \rangle_p$ is a nonzero proper p-ideal of L (see [30, Exercise 2.1.2 and Proposition 2.1.3(4)]). By induction hypothesis, I and L/I have maximal 0-PIM, and therefore it follows from Lemma 2.1 that L also has maximal 0-PIM. □

Remark. It is easy to see directly that one-dimensional restricted Lie algebras have maximal 0-PIM. As in the proof of Theorem 6.4 one could assume in the proof of Theorem 2.2 that the ground field is algebraically closed and then one could use in the induction step that finite-dimensional solvable restricted Lie algebras over algebraically closed fields have a p-ideal of codimension one.

According to the conjugacy of maximal tori due to David John Winter, any maximal torus of a finite-dimensional solvable restricted Lie algebra L has dimension $\mathrm{MT}(L)$ (see [34, Proposition 2.17] or also [26, Theorem 1.5.6]). As a consequence of this in conjunction with Theorem 2.2, one obtains the following generalization of [6, Satz II.3.2], [8, Corollary 1], [3, Corollary 4.5], [4, Proposition 2.2], and [10, Proposition 1] to solvable restricted Lie algebras:

Corollary 2.3. *Let L be a finite-dimensional solvable restricted Lie algebra over a field $\mathbb{F}$ of prime characteristic p, and let χ be a linear form on L. If S is a one-dimensional L-module with p-character χ, then $P_L(S) \cong \mathrm{Ind}_T^L(S_{|T}, \chi)$ holds for every maximal torus T of L.*

Proof. As a consequence of Winter's theorem on the conjugacy of maximal tori, any maximal torus T of L is a torus of maximal dimension $\mathrm{MT}(L)$ (see [34, Proposition 2.17] or [26, Theorem 1.5.6]). Now it follows from [11, Lemma 1] and Theorem 2.2 that

$$\dim_{\mathbb{F}} P_L(S) = \dim_{\mathbb{F}} P_L(\mathbb{F}) = p^{\dim_{\mathbb{F}} L - \mathrm{MT}(L)},$$

and the assertion is an immediate consequence of Corollary 1.3. □

3. An upper bound for the number of irreducible modules

As an application of Theorem 2.2 one obtains an upper bound for the number of the isomorphism classes of irreducible modules with a fixed p-character for a solvable restricted Lie algebra. This result generalizes [11, Theorem 4] and at the same time simplifies the proof considerably.

Theorem 3.1. *Let L be a finite-dimensional solvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ of prime characteristic p, and let χ be a linear form on L. Then the number of isomorphism classes of irreducible L-modules with p-character χ is at most $p^{\mathrm{MT}(L)}$.*

Proof. Let $\mathrm{Irr}(L, \chi)$ denote the set of isomorphism classes of irreducible L-modules with p-character χ, and let $|\mathrm{Irr}(L, \chi)|$ denote its cardinality. It follows from Proposition 1.4 and Theorem 2.2 that

$$\begin{aligned} p^{\dim_{\mathbb{F}} L} = \dim_{\mathbb{F}} u(L,\chi) &= \sum_{[S]\in\mathrm{Irr}(L,\chi)} (\dim_{\mathbb{F}} S)\cdot[\dim_{\mathbb{F}} P_L(S)] \\ &= \sum_{[S]\in\mathrm{Irr}(L,\chi)} \dim_{\mathbb{F}}[P_L(S)\otimes S^*] \\ &\geq |\mathrm{Irr}(L,\chi)|\cdot\dim_{\mathbb{F}} P_L(\mathbb{F}) \\ &= |\mathrm{Irr}(L,\chi)|\cdot p^{\dim_{\mathbb{F}} L-\mathrm{MT}(L)}\,. \end{aligned}$$

Dividing both sides of the inequality by $p^{\dim_{\mathbb{F}} L-\mathrm{MT}(L)}$ yields $|\mathrm{Irr}(L,\chi)|\leq p^{\mathrm{MT}(L)}$, and thus the claim. □

Remark. If one uses [8, Theorem 2] instead of Proposition 1.4, then the proof of Theorem 3.1 yields a new proof of [24, Satz 6]. It should be mentioned that the irreducible modules for a finite-dimensional solvable restricted Lie algebra over an algebraically closed field of characteristic $p>2$ have already been described by Helmut Strade in [24, Satz 3 and 4]. The authors were informed by Sasha Premet that using this result one may obtain another proof of Theorem 3.1 for $p>2$, which, however, would be more involved than the one given here.

Of course, the proof of Theorem 3.1 holds for any restricted Lie algebra with maximal 0-PIM. However, it will be shown in the last section that, at least in characteristic $p>3$, such Lie algebras are necessarily solvable.

4. Induced representations of filtered restricted Lie algebras

Let L be a restricted Lie algebra over a field of prime characteristic p with a descending restricted filtration

$$L=L_{(-d)}\supset L_{(-d+1)}\supset L_{(-d+2)}\supset\cdots\supset L_{(h)}\supset L_{(h+1)}=0 \tag{4.1}$$

of depth d and height h. For integers $n>h$ put $L_{(n)}:=0$ and for integers $m<-d$ put $L_{(m)}:=L$. Recall that $[L_{(m)},L_{(n)}]\subseteq L_{(m+n)}$ for all $m,n\in\mathbb{Z}$ and $L_{(n)}^{[p]}\subseteq L_{(pn)}$ for every $n\in\mathbb{Z}$ (see [30, Section 1.9 and Section 3.1]. In particular, $L_{(n)}$ is a p-subalgebra of L and a p-ideal of $L_{(0)}$ for every non-negative integer n. Moreover, $L_{(n)}^{[p]^k}=0$ for every positive integer n and $k\gg 0$.

The graded vector space $\mathrm{gr}(L):=\bigoplus_{n\in\mathbb{Z}}\mathrm{gr}_n(L)$, where $\mathrm{gr}_n(L):=L_{(n)}/L_{(n+1)}$, associated with the filtration $(L_{(n)})_{n\in\mathbb{Z}}$ carries canonically the structure of a graded restricted Lie algebra with bracket $[\cdot,\cdot]\colon\mathrm{gr}(L)\times\mathrm{gr}(L)\to\mathrm{gr}(L)$ given by

$$[x+L_{(m+1)},y+L_{(n+1)}]=[x,y]+L_{(m+n+1)}\qquad\text{for all } m,n\in\mathbb{Z}, x\in L_{(m)}, y\in L_{(n)}$$

and $[p]$-mapping $(\cdot)^{[p]}\colon\mathrm{gr}(L)\to\mathrm{gr}(L)$ given by

$$(x+L_{(n+1)})^{[p]}=x^{[p]}+L_{(pn+1)}\quad\text{for every } n\in\mathbb{Z}\text{ and for every } x\in L_{(n)}$$

(see [30, Theorem 3.3.1]).

Any descending filtration $(L_{(n)})_{n\in\mathbb{Z}}$ of a Lie algebra L induces a descending filtration $(U(L)_{(n)})_{n\in\mathbb{Z}}$ on the universal enveloping algebra $U(L)$ of L, where

$$U(L)_{(n)} := \sum_{\substack{s\geq 0,\ n_j\geq -d\\ n_1+\cdots+n_s\geq n}} L_{(n_1)}\cdots L_{(n_s)}\,. \tag{4.2}$$

Moreover, if $\mathrm{gr}(U(L)) := \bigoplus_{n\in\mathbb{Z}} \mathrm{gr}_n(U(L))$, where $\mathrm{gr}_n(U(L)) := U(L)_{(n)}/U(L)_{(n+1)}$ for every $n \in \mathbb{Z}$, denotes the associated graded algebra, one has a canonical isomorphism $\phi_\bullet\colon U(\mathrm{gr}(L)) \to \mathrm{gr}(U(L))$ induced by $\phi(x + L_{(n+1)}) := x + U(L)_{(n+1)}$ for every $n \in \mathbb{Z}$ and every $x \in L_{(n)}$ (cf. [30, Theorem 1.9.5] for ascending filtrations which carries over verbatim to descending filtrations).

Let $u(L) := u(L, 0)$ denote the restricted universal enveloping algebra of a restricted Lie algebra L. In particular, one has a surjective homomorphism of algebras $\pi\colon U(L) \to u(L)$. Then $u(L)$ carries the filtration $(u(L)_{(n)})_{n\in\mathbb{Z}}$ given by $u(L)_{(n)} := \pi(U(L)_{(n)})$ for every $n \in \mathbb{Z}$. Thus, if $\mathrm{gr}(u(L)) := \bigoplus_{n\in\mathbb{Z}} \mathrm{gr}_n(u(L))$, where $\mathrm{gr}_n(u(L)) := u(L)_{(n)}/u(L)_{(n+1)}$ for every $n \in \mathbb{Z}$, denotes the graded algebra associated with the filtration $(u(L)_{(n)})_{n\in\mathbb{Z}}$, by construction, one has a canonical surjective homomorphism of graded algebras $\mathrm{gr}(\pi)\colon \mathrm{gr}(U(L)) \to \mathrm{gr}(u(L))$.

Suppose now that $(L_n)_{n\in\mathbb{Z}}$ is a restricted filtration of the restricted Lie algebra L as defined in (4.1). Then $\mathrm{gr}(L)$ is a graded restricted Lie algebra, and one has a surjective homomorphism of graded algebras $\pi_{\mathrm{gr}}\colon U(\mathrm{gr}(L)) \to u(\mathrm{gr}(L))$ with respect to the total degree. By construction, one has a homomorphism of graded restricted Lie algebras $\varphi\colon \mathrm{gr}(L) \to \mathrm{gr}(u(L))$, $\varphi(x + L_{(n+1)}) := x + u(L)_{(n+1)}$, satisfying $\varphi((x + L_{(n+1)})^{[p]}) = \varphi(x + L_{(n+1)})^p$ for every $n \in \mathbb{Z}$ and every $x \in L_{(n)}$. Hence φ induces a homomorphism of graded algebras $\varphi_\bullet\colon u(\mathrm{gr}(L)) \to \mathrm{gr}(u(L))$, i.e., one has a commutative diagram

$$\begin{array}{ccc} U(\mathrm{gr}(L)) & \xrightarrow{\ \phi_\bullet\ } & \mathrm{gr}(U(L)) \\ {\scriptstyle \pi_{\mathrm{gr}}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}(\pi)} \\ u(\mathrm{gr}(L)) & \xrightarrow{\ \varphi_\bullet\ } & \mathrm{gr}(u(L)) \end{array}$$

In particular, $\varphi_\bullet$ is surjective. The following result will be important for our purpose.

Proposition 4.1. *If $(L_{(n)})_{n\in\mathbb{Z}}$ is a descending restricted filtration of a restricted Lie algebra L of depth d and height h, then $\varphi_\bullet\colon u(\mathrm{gr}(L)) \to \mathrm{gr}(u(L))$ is an isomorphism of graded algebras.*

Proof. Let $X = \bigsqcup_{-d\leq n\leq h} X_n$ be an ordered basis of L such that

(i) for $x \in X_m$ and $y \in X_n$ with $m < n$ one has $x < y$,
(ii) $\{x + L_{(n+1)} \mid x \in X_n\}$ is a basis of $\mathrm{gr}_n(L)$.

Let $\mathfrak{F}$ denote the set of functions from X to $\{0, \ldots, p-1\}$ with finite support. For $\alpha \in \mathfrak{F}$ let $x^\alpha \in u(L)$ denote the monomial $\prod_{x\in X} x^{\alpha(x)}$, where the product is taken with respect to increasing order. For $\alpha \in \mathfrak{F}$ put

$$|\alpha| := \sum_{-d\leq n\leq h} n \cdot \sum_{x\in X_n} \alpha(x) \in \mathbb{Z}\,.$$

Then, by [30, Proposition 1.9.1] and Jacobson's analogue of the Poincaré-Birkhoff-Witt theorem (see [30, Theorem 2.5.1(2)]) $\{x^\alpha + u(L)_{(n+1)} \mid |\alpha| = n\}$ is a basis of $\mathrm{gr}_n(u(L))$, and $\{\prod_{-d\le n\le h}\prod_{x\in X_n}(x + L_{(n+1)})^{\alpha(x)} \mid |\alpha| = n\}$, where products are taken with respect to increasing order in X, is a basis of the homogeneous component of $u(\mathrm{gr}(L))$ of total degree n. Hence, since $\varphi_\bullet$ is mapping a basis injectively onto a basis, $\varphi_\bullet$ is an isomorphism. □

From the proof of Proposition 4.1 and (4.2) one deduces the following properties:

Corollary 4.2. *If $(L_{(n)})_{n\in\mathbb{Z}}$ is a descending restricted filtration of a restricted Lie algebra L of depth d and height h, then the following statements hold:*

(1) $u(L)_{(1)} \subseteq u(L)\cdot\omega(L_{(1)})$, *where $\omega(L_{(1)})$ denotes the augmentation ideal of $u(L_{(1)})$.*
(2) $u(L)_{(0)} \subseteq u(L_{(0)}) + u(L)\cdot\omega(L_{(1)})$.

Let $(L_{(n)})_{n\in\mathbb{Z}}$ be a descending restricted filtration of a restricted Lie algebra L of depth d and height h. Then every restricted $\mathrm{gr}_0(L)$-module V can be considered as a restricted $L_{(0)}$-module. The induced module

$$M := \mathrm{Ind}_{L_{(0)}}^{L}(V,0) := u(L)\otimes_{u(L_{(0)})} V$$

has a descending filtration $(M_{(r)})_{r\in\mathbb{Z}}$ given by $M_{(r)} := u(L)_{(r)}(1\otimes V)$, i.e., for all $r,s\in\mathbb{Z}$ one has

$$u(L)_{(r)}\cdot M_{(s)} \subseteq M_{(r+s)}\,. \tag{4.3}$$

Since the restricted $L_{(0)}$-module V is inflated from $\mathrm{gr}_0(L)$, Corollary 4.2 implies that $M_{(0)} = 1\otimes V$, and $M_{(r)} = 0$ for every integer $r > 0$.

Let $\mathrm{gr}(M) := \bigoplus_{r\in\mathbb{Z}}\mathrm{gr}_r(M)$, where $\mathrm{gr}_r(M) := M_{(r)}/M_{(r+1)}$, denote the associated graded vector space. According to (4.3), $\mathrm{gr}(M)$ is a $\mathrm{gr}(u(L))$-module. Thus, by Proposition 4.1, $\mathrm{gr}(M)$ may be considered as a restricted $\mathrm{gr}(L)$-module.

Theorem 4.3. *Let L be a restricted Lie algebra, and let $(L_{(n)})_{n\in\mathbb{Z}}$ be a descending restricted filtration of L of depth d and height h. Let V be a restricted $\mathrm{gr}_0(L)$-module, and let $M := \mathrm{Ind}_{L_{(0)}}^{L}(V,0)$. Then one has a canonical isomorphism of graded restricted $\mathrm{gr}(L)$-modules*

$$\mathrm{Ind}_{\mathrm{gr}_+(L)}^{\mathrm{gr}(L)}(V,0) \cong \mathrm{gr}(M)\,,$$

where $\mathrm{gr}_+(L) := \bigoplus_{n\ge 0}\mathrm{gr}_n(L)$. In particular, if $\mathrm{Ind}_{\mathrm{gr}_+(L)}^{\mathrm{gr}(L)}(V,0)$ is an irreducible $\mathrm{gr}(L)$-module, then M is an irreducible L-module.

Proof. It follows from $M_{(0)} = 1\otimes V$and $M_{(r)} = 0$ for every integer $r > 0$ that one has an isomorphism $\iota\colon V \to \mathrm{gr}_0(M)$ of graded $u(\mathrm{gr}_+(L))$-modules. As the induction functor is left adjoint to the restriction functor, this yields a mapping of graded $u(\mathrm{gr}(L))$-modules $\iota_\bullet\colon \mathrm{Ind}_{\mathrm{gr}_+(L)}^{\mathrm{gr}(L)}(V,0) \to \mathrm{gr}(M)$. By construction, $\iota_\bullet$ is surjective.

Let $X' = \bigsqcup_{-d\le n\le -1} X_n$ be an ordered subset of L, where X_n has the same properties as in the proof of Proposition 4.1, and let $\mathfrak{F}'$ denote the set of functions from X' to $\{0,\dots,p-1\}$ with finite support. Then, using similar notations as in the proof of Proposition 4.1, one obtains for any $r<0$ an isomorphism of vector spaces

$$\mathrm{gr}_r(M)\cong\bigoplus_{\alpha\in\mathfrak{F}',|\alpha|=r}\mathbb{F}x^\alpha\otimes V\,.$$

By Jacobson's analogue of the Poincaré-Birkhoff-Witt theorem (see [30, Theorem 2.5.1(2)]), one has an isomorphism of graded right $u(\mathrm{gr}_+(L))$-modules

$$u(\mathrm{gr}(L))\cong\bigoplus_{\alpha\in\mathfrak{F}'}\mathbb{F}x^\alpha\otimes u(\mathrm{gr}_+(L))$$

with respect to the total degree. This shows that $\iota_\bullet$ is an isomorphism. The final remark is a direct consequence of the fact that any non-zero proper L-submodule S of M gives rise to a non-zero proper $\mathrm{gr}(L)$-module $\mathrm{gr}(S)$ of $\mathrm{gr}(M)$, where one puts $\mathrm{gr}_r(S):=(S\cap M_{(r)})+M_{(r+1)}/M_{(r+1)}$ for every $r\in\mathbb{Z}$. □

5. Representations of non-graded Hamiltonian Lie algebras

Let $\mathcal{O}(2;\underline{1})=\mathbb{F}[x,y]/\mathbb{F}[x,y]\{x^p,y^p\}$ denote the truncated polynomial algebra in two variables x and y over a field $\mathbb{F}$ of prime characteristic p. By $\mathcal{O}(2;\underline{1})^{(1)}$ we denote its augmentation ideal, i.e., $f\in\mathcal{O}(2;\underline{1})^{(1)}$ if, and only if, f has zero constant term. Moreover, $\partial_x=\partial/\partial x$ and $\partial_y=\partial/\partial y$ will denote the partial derivatives of the algebra $\mathcal{O}(2;\underline{1})$.

The Hamiltonian Lie algebra $H(2;\underline{1})$ is defined as the subalgebra of the Witt-Jacobson algebra $W(2;\underline{1})$ whose elements annihilate the 2-form $\omega_H:=\mathrm{dx}\wedge\mathrm{dy}$, i.e., one has

$$\begin{aligned}H(2;\underline{1})&:=\{D\in W(2;\underline{1})\mid D(\omega_H)=0\} &(5.1)\\ &=\{f\partial_x+g\partial_y\in W(2;\underline{1})\mid \partial_y(g)=-\partial_x(f)\}\,. &(5.2)\end{aligned}$$

Since $W(2;\underline{1})$ is a graded restricted Lie algebra, (5.1) implies that $H(2;\underline{1})$ is also a graded restricted Lie algebra. Let $\{\cdot,\cdot\}\colon\mathcal{O}(2;\underline{1})\times\mathcal{O}(2;\underline{1})\to\mathcal{O}(2,\underline{1})$ denote the standard Poisson bracket given by

$$\{f,g\}=(\partial_x f)(\partial_y g)-(\partial_y f)(\partial_x g)\quad\text{for all } f,g\in\mathcal{O}(2;\underline{1})\,. \tag{5.3}$$

Then $(\mathcal{O}(2;\underline{1}),\{\cdot,\cdot\})$ is a Lie algebra, and the canonical map (of degree -2)

$$\mathbf{D}\colon(\mathcal{O}(2;\underline{1}),\{\cdot,\cdot\})\to W(2;\underline{1})\,,\quad \mathbf{D}(f):=\partial_x(f)\partial_y-\partial_y(f)\partial_x\quad\text{for } f\in\mathcal{O}(2;\underline{1})\,, \tag{5.4}$$

defines a Lie algebra homomorphism satisfying $\mathrm{Ker}(\mathbf{D})=\mathbb{F}1$ and $\mathrm{Im}(\mathbf{D})\subseteq H(2;\underline{1})$. It is well known that $\mathrm{Im}(\mathbf{D})$ is an ideal in $H(2;\underline{1})$ containing the derived subalgebra of $H(2;\underline{1})$. Moreover, for $p\geq 3$

$$H(2;\underline{1})^{(2)}=\langle\{\mathbf{D}(x^ay^b)\mid 0<a+b<2(p-1)\}\rangle_{\mathbb{F}}\subset H(2;\underline{1}) \tag{5.5}$$

is a simple ideal of $H(2;\underline{1})$. For further details see [26, §4.2] and [30, §4.4]. (Note that in [30], the algebra $H(2;\underline{1})$ is denoted by $H''(2;\underline{1})$, and that $\mathrm{Im}(\mathbf{D})$ is denoted by $H'(2;\underline{1})$.)

Define $\Gamma := x^{p-1}\partial_y$ and $\Theta := -y^{p-1}\partial_x$. Then, by (5.2), $\Gamma, \Theta \in H(2;\underline{1})$. Since $x^{p-1} \notin \mathrm{Im}(\partial_x)$ and $y^{p-1} \notin \mathrm{Im}(\partial_y)$, (5.4) implies that $\Gamma, \Theta \notin \mathrm{Im}(\mathbf{D})$. Hence

$$\mathbf{Y} := \mathrm{Im}(\mathbf{D}) \oplus \mathbb{F}\Gamma \oplus \mathbb{F}\Theta \subseteq H(2;\underline{1}) \tag{5.6}$$

is a graded subalgebra of $H(2;\underline{1})$ of dimension p^2+1. (Indeed, it follows from [1, Proposition 2.1.8(b)(ii)] that $\dim_{\mathbb{F}} H(2;\underline{1}) = p^2+1$, so $\mathbf{Y} = H(2;\underline{1})$, but this is irrelevant for our purpose.) From the identities

$$[\Gamma, \mathbf{D}(x^a y^b)] = b\,\mathbf{D}(x^{p-1+a}y^{b-1})\,, \tag{5.7}$$

$$[\Theta, \mathbf{D}(x^a y^b)] = -a\,\mathbf{D}(x^{a-1}y^{p-1+b})\,, \tag{5.8}$$

$$[\Gamma, \Theta] = \mathbf{D}(x^{p-1}y^{p-1})\,, \tag{5.9}$$

$$[\mathbf{D}(x), \mathbf{D}(x^a y^b)] = b\,\mathbf{D}(x^a y^{b-1})\,, \tag{5.10}$$

$$[\mathbf{D}(y), \mathbf{D}(x^a y^b)] = -a\,\mathbf{D}(x^{a-1}y^b)\,, \tag{5.11}$$

$$\mathbf{D}(x^a y^b)^{[p]} = \begin{cases} 0 & \text{if } (a,b) \neq (1,1)\,, \\ \mathbf{D}(xy) & \text{if } a = b = 1\,, \end{cases} \tag{5.12}$$

$$\Gamma^{[p]} = 0\,, \tag{5.13}$$

$$\Theta^{[p]} = 0\,. \tag{5.14}$$

for all $a, b \in \{0, \dots, p-1\}$ with $(a,b) \neq (0,0)$, one concludes that $\mathbf{Y}$ is a graded p-subalgebra of $H(2;\underline{1})$. Moreover, by construction, one has the following property:

Proposition 5.1. *The graded vector space* $\mathrm{Coker}(H(2;\underline{1})^{(2)} \to \mathbf{Y})$ *is concentrated in degrees* $p-2$ *and* $2p-4$.

Remark. Note that $\mathbf{Y}_0 := \mathbf{Y} \cap H(2;\underline{1})_0 \cong \mathfrak{sl}_2(\mathbb{F})$ (see [30, Proposition 4.4.4(4)]).

The Hamiltonian Lie algebra $H(2;\underline{1};\Phi(\tau))$ is defined as the subalgebra of the Witt-Jacobson algebra $W(2;\underline{1})$ whose elements annihilate the 2-form

$$\omega_{\Phi(\tau)} := (1 + x^{p-1}y^{p-1})\,\mathrm{dx} \wedge \mathrm{dy}\,,$$

i.e., one has

$$H(2;\underline{1};\Phi(\tau)) := \{D \in W(2;\underline{1}) \mid D(\omega_{\Phi(\tau)}) = 0\}\,. \tag{5.15}$$

Thus, $H(2;\underline{1};\Phi(\tau))$ is a p-subalgebra of $W(2;\underline{1})$. The element $f\partial_x + g\partial_x \in W(2;\underline{1})$ is contained in $H(2;\underline{1};\Phi(\tau))$ if, and only if,

$$(1 + x^{p-1}y^{p-1})(\partial_x f + \partial_y g) - x^{p-2}y^{p-2}(yf + xg) = 0\,. \tag{5.16}$$

For the distinguished element $\Lambda = 1 - x^{p-1}y^{p-1} \in \mathcal{O}(2;\underline{1})$ one defines the Poisson bracket $\{\cdot,\cdot\}_\Lambda \colon \mathcal{O}(2;\underline{1}) \times \mathcal{O}(2;\underline{1}) \to \mathcal{O}(2;\underline{1})$ by

$$\{f, g\}_\Lambda = \big((\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)\big)\Lambda \quad \text{for all } f, g \in \mathcal{O}(2,\underline{1})\,. \tag{5.17}$$

Then $(\mathcal{O}(2;\underline{1}), \{\cdot,\cdot\}_\Lambda)$ is a Lie algebra, and the map $\mathbf{D}_\Lambda = \Lambda{\cdot}\mathbf{D}\colon \mathcal{O}(2;\underline{1}) \to W(2;\underline{1})$, where $\mathbf{D}$ is defined as in (5.4), is a homomorphism of Lie algebras satisfying $\mathrm{Ker}(\mathbf{D}_\Lambda) = \mathbb{F}1$ and $H := \mathrm{Im}(\mathbf{D}_\Lambda) \subseteq H(2;\underline{1};\Phi(\tau))$. Moreover, one has

$$\{(a,b) \mid 0 \le a,b \le p-1,\ \mathbf{D}_\Lambda(x^a y^b) \ne \mathbf{D}(x^a y^b)\,\} = \{(1,0),(0,1)\}\,. \tag{5.18}$$

For $p > 3$ the Lie algebra H coincides with the derived subalgebra $H(2;\underline{1};\Phi(\tau))^{(1)}$ of $H(2;\underline{1};\Phi(\tau))$ which is a simple Lie algebra. For further details the reader may wish to consult [25, §1] and the references therein, as well as [26, §6.3].

Let $\Gamma := x^{p-1}\partial_y$, $\Theta := -y^{p-1}\partial_x \in W(2;\underline{1})$ be given as before. Then, by (5.16), $\Gamma, \Theta \in H(2;\underline{1};\Phi(\tau))$, but $\Gamma, \Theta \notin H$, and

$$L := H \oplus \mathbb{F}\Gamma \oplus \mathbb{F}\Theta\,, \tag{5.19}$$

is a p-subalgebra of $H(2;\underline{1};\Phi(\tau))$ that is isomorphic to $\mathrm{Der}(H)$ (cf. [25, Proposition 1.1(1)]). Hence, for $p > 3$, it follows from [28, Theorem 10.3.1] and the argument in the middle of [28, p. 41] that L coincides with the minimal p-envelope of H in $W(2;\underline{1})$. The following identities hold in L (see [25, p. 582]):

$$[\Gamma, \mathbf{D}_\Lambda(x^a y^b)] = b\,\mathbf{D}_\Lambda(x^{p-1+a} y^{b-1})\,, \tag{5.20}$$

$$[\Theta, \mathbf{D}_\Lambda(x^a y^b)] = -a\,\mathbf{D}_\Lambda(x^{a-1} y^{p-1+b})\,, \tag{5.21}$$

$$[\Gamma, \Theta] = \mathbf{D}_\Lambda(x^{p-1} y^{p-1})\,, \tag{5.22}$$

$$[\mathbf{D}_\Lambda(x), \mathbf{D}_\Lambda(x^a y^b)] = \begin{cases} b\,\mathbf{D}_\Lambda(x^a y^{b-1}) & \text{if } (a,b) \ne (0,1)\,, \\ -\mathbf{D}_\Lambda(x^{p-1} y^{p-1}) & \text{if } (a,b) = (0,1)\,, \end{cases} \tag{5.23}$$

$$[\mathbf{D}_\Lambda(y), \mathbf{D}_\Lambda(x^a y^b)] = \begin{cases} -a\,\mathbf{D}_\Lambda(x^{a-1} y^b) & \text{if } (a,b) \ne (1,0)\,, \\ \mathbf{D}_\Lambda(x^{p-1} y^{p-1}) & \text{if } (a,b) = (1,0)\,, \end{cases} \tag{5.24}$$

$$\mathbf{D}_\Lambda(x^a y^b)^{[p]} = \begin{cases} 0 & \text{if } (a,b) \notin \{(0,1),(1,0),(1,1)\}\,, \\ \mathbf{D}_\Lambda(xy) & \text{if } a = b = 1\,, \\ \Gamma & \text{if } (a,b) = (1,0)\,, \\ \Theta & \text{if } (a,b) = (0,1)\,, \end{cases} \tag{5.25}$$

$$\Gamma^{[p]} = 0, \tag{5.26}$$

$$\Theta^{[p]} = 0\,. \tag{5.27}$$

The natural grading on $W(2;\underline{1})$ induces filtrations

$$H_{(n)} := H \cap W(2;\underline{1})_{(n)}, \qquad L_{(n)} := L \cap W(2;\underline{1})_{(n)} \ \text{ for every } n \in \mathbb{Z}\,, \tag{5.28}$$

where $W(2;\underline{1})_{(n)} = \bigoplus_{d \ge n} W(2;\underline{1})_d$. The following result will be important for our purpose.

Proposition 5.2. *The family $(L_{(n)})_{n\in\mathbb{Z}}$ is a descending restricted filtration of L of depth* 1 *and height* $2p-4$. *Moreover,* $\mathrm{gr}(L) \cong \mathbf{Y}$.

Proof. According to [25, Proposition 1.2(1)], one has that $L = L_{(-1)}$, $H = H_{(-1)}$, $H_{(n)} = L_{(n)} = 0$ for $n > 2p-4$, and for $-1 \leq n \leq 2p-4$:

$$H_{(n)} = \mathbb{F}\mathbf{D}_\Lambda(x^{p-1}y^{p-1}) + \langle\{\mathbf{D}_\Lambda(x^a y^b) \mid n+2 \leq a+b \leq 2p-3\}\rangle_{\mathbb{F}}\,,$$
$$L_{(n)} = \begin{cases} H_{(n)} \oplus \mathbb{F}\Gamma \oplus \mathbb{F}\Theta & \text{for } -1 \leq n \leq p-2\,, \\ \quad H_{(n)} & \text{for } p-2 < n \leq 2p-4\,. \end{cases} \tag{5.29}$$

From the identities (5.25)–(5.27) one obtains that $L_{(n)}^{[p]} \subseteq L_{(pn)}$ for every integer $n \geq 0$. In the associated graded algebra $\operatorname{gr}(L)$ one has the additional identity (see (5.24))

$$[\mathbf{D}_\Lambda(x) + L_{(0)}, \mathbf{D}_\Lambda(y) + L_{(0)}] = 0\,. \tag{5.30}$$

Hence using (5.30) and the identities (5.20)–(5.24) one concludes that the linear map $\psi\colon \operatorname{gr}(L) \to \mathbf{Y}$ given by

$$\begin{aligned} \psi(\mathbf{D}_\Lambda(x^a y^b) + L_{(a+b-1)}) &:= \mathbf{D}(x^a y^b)\,, \\ \psi(\Gamma + L_{(p-1)}) &:= \Gamma\,, \\ \psi(\Theta + L_{(p-1)}) &:= \Theta\,, \end{aligned} \tag{5.31}$$

is an isomorphism of graded Lie algebras. The identities (5.25)–(5.27) imply that the mapping $\psi|_{\operatorname{gr}_+(L)}\colon \operatorname{gr}_+(L) \to \mathbf{Y}$ is a homomorphism of restricted Lie algebras. Thus, as $\mathbf{Y}$ is a graded restricted Lie algebra, the filtration $(L_{(n)})_{n\in\mathbb{Z}}$ is restricted. This yields the claim. □

It is well known that $L_{(0)}/L_{(1)} \cong \mathbf{Y}_0 \cong \mathfrak{sl}_2(\mathbb{F})$ (see [25, Proposition 1.2(2.b)] and the remark after Proposition 5.1). For $\lambda \in \{0, \dots, p-1\}$ let $V(\lambda)$ denote the irreducible restricted $\mathfrak{sl}_2(\mathbb{F})$-module of highest weight λ, i.e., $\dim_{\mathbb{F}} V(\lambda) = \lambda + 1$. By $V(\lambda)$ we will also denote its inflation to $L_{(0)}$. The following result will be important for the proof of the main result in the next section.

Theorem 5.3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 3$. Then the restricted L-module $M(\lambda) := \operatorname{Ind}_{L_{(0)}}^{L}(V(\lambda), 0)$ is irreducible for every $\lambda \in \{2, \dots, p-1\}$.*

Proof. By virtue of Proposition 5.2, $(L_{(n)})_{n\in\mathbb{Z}}$ is a restricted filtration of L and $\operatorname{gr}(L) \cong \mathbf{Y}$. Put $\mathbf{X} := H(2;\underline{1})^{(2)} \subseteq \mathbf{Y}$, and set $\mathbf{X}_{(n)} := \mathbf{X} \cap \mathbf{Y}_{(n)}$ for any $n \in \mathbb{Z}$. According to Proposition 5.1, $\operatorname{Coker}(\mathbf{X} \to \mathbf{Y})$ is concentrated in degrees $p-2$ and $2p-4$. Hence the canonical map

$$\operatorname{Ind}_{\mathbf{X}_{(0)}}^{\mathbf{X}}(V(\lambda)_{|\mathbf{X}_{(0)}}, 0) \longrightarrow \operatorname{Ind}_{\mathbf{Y}_{(0)}}^{\mathbf{Y}}(V(\lambda), 0)_{|\mathbf{X}} \tag{5.32}$$

is an isomorphism. As $\operatorname{Ind}_{\mathbf{X}_{(0)}}^{\mathbf{X}}(V(\lambda)_{|\mathbf{X}_{(0)}}, 0)$ is an irreducible restricted $\mathbf{X}$-module for $\lambda \neq 0, 1$ (see [13, p. 253]), this implies that $\operatorname{Ind}_{\mathbf{Y}_{(0)}}^{\mathbf{Y}}(V(\lambda), 0)$ is an irreducible $\mathbf{Y}$-module. Consequently, by Theorem 4.3, $M(\lambda)$ is an irreducible L-module unless $\lambda = 0, 1$. □

Remark. Since the induced modules in Theorem 5.3 coincide with the corresponding coinduced modules, and as Shen's mixed products are isomorphic to coinduced modules (see [2, Corollary 2.6] and [5, Proposition 1.5(1)]), the irreducibility of $\operatorname{Ind}_{\mathbf{X}_{(0)}}^{\mathbf{X}}(V(\lambda)_{|\mathbf{X}_{(0)}}, 0)$ in the proof of Theorem 5.3 follows also from [23, Proposition 1.2].

6. Restricted Lie algebras with maximal 0-PIM

The purpose of this section is to prove that for fields of characteristic $p > 3$ finite-dimensional restricted Lie algebras having maximal 0-PIM are necessarily solvable. The following criterion for a restricted Lie algebra having maximal 0-PIM will be very useful in the proof of this result. For a Lie algebra L and any L-module M we denote by $M^L := \{m \in M \mid \forall\, x \in L : x \cdot m = 0\}$ the *space of L-invariants* of M.

Lemma 6.1. *Let L be a finite-dimensional restricted Lie algebra over a field of prime characteristic, and let T be a torus in L of maximal dimension. Then L has maximal 0-PIM if, and only if, $S^T = 0$ for every non-trivial irreducible restricted L-module S.*

Proof. Suppose first that L has maximal 0-PIM. Then it follows from Corollary 1.3 that $P_L(\mathbb{F}) \cong \mathrm{Ind}_T^L(\mathbb{F}, 0)$. Since $\mathrm{Hom}_L(P_L(\mathbb{F}), S) = 0$ for every non-trivial irreducible restricted L-module S, Frobenius reciprocity yields:

$$S^T \cong \mathrm{Hom}_T(\mathbb{F}, S_{|T}) \cong \mathrm{Hom}_L(\mathrm{Ind}_T^L(\mathbb{F}, 0), S) \cong \mathrm{Hom}_L(P_L(\mathbb{F}), S) = 0 \qquad (6.1)$$

for every non-trivial irreducible restricted L-module S.

Conversely, suppose that $S^T = 0$ for every non-trivial irreducible restricted L-module S. Reading (6.1) backwards shows that the head of $\mathrm{Ind}_T^L(\mathbb{F}, 0)$ is a trivial L-module. On the other hand, applying Frobenius reciprocity to the one-dimensional trivial irreducible L-module, one has

$$\mathrm{Hom}_L(\mathrm{Ind}_T^L(\mathbb{F}, 0), \mathbb{F}) \cong \mathrm{Hom}_T(\mathbb{F}, \mathbb{F}) \cong \mathbb{F}\,.$$

This implies that the head of $\mathrm{Ind}_T^L(\mathbb{F}, 0)$ is one-dimensional, and therefore $\mathrm{Ind}_T^L(\mathbb{F}, 0)$ is indecomposable. By virtue of Proposition 1.2, $P_L(\mathbb{F})$ is a direct summand of $\mathrm{Ind}_T^L(\mathbb{F}, 0)$. As a consequence one obtains that $P_L(\mathbb{F}) \cong \mathrm{Ind}_T^L(\mathbb{F}, 0)$. Hence L has maximal 0-PIM. □

The next result is an immediate consequence of Theorem 2.2 and Lemma 6.1, but we will give an independent proof. By virtue of Lemma 6.1, this then would provide an alternative proof of Theorem 2.2.

Lemma 6.2. *Let L be a finite-dimensional solvable restricted Lie algebra over a field of prime characteristic, let T be a torus in L of maximal dimension, and let S be a non-trivial irreducible restricted L-module. Then $S^T = 0$.*

Proof. It follows from the conjugacy of maximal tori due to David John Winter that every maximal torus of L has maximal dimension (see [34, Proposition 2.17] or [26, Theorem 1.5.6]). Consequently, tori in L of maximal dimension are the same as maximal tori of L. According to [30, Theorem 2.4.5(1)], it is enough to prove the assertion for $L/\mathrm{Ann}_L(S)$, where $\mathrm{Ann}_L(S) := \{x \in L \mid x \cdot s = 0 \text{ for every } s \in S\}$ is the annihilator of S in L. So we may assume without loss of generality that $\mathrm{Ann}_L(S) = 0$.

Now consider any non-zero abelian p-ideal A of L. Then the maximal torus T_0 of A is non-zero since otherwise [30, Theorem 2.3.4] in conjunction with the

Engel-Jacobson Theorem [30, Theorem 1.3.1] would imply that $A \subseteq \mathrm{Ann}_L(S) = 0$, which is a contradiction. As A is an abelian ideal of L, A acts nilpotently on L. Hence T_0 acts trivially on L, and thus T_0 is central in L. In particular, T_0 is contained in T, and therefore it is enough to show that $S^{T_0} = 0$. But this is now clear because T_0 is central in L and S is an irreducible L-module. □

The following result, which will be used in the proof of Theorem 6.4, is included here for completeness.

Lemma 6.3. *Let L be a finite-dimensional non-abelian restricted Lie algebra over a field of prime characteristic p with no non-zero proper p-ideals. Then the following statements hold:*

(1) *$[L, L]$ is the unique minimal ideal of L.*
(2) *L is a minimal p-envelope of $[L, L]$.*
(3) *$[L, L]$ is simple as an ordinary Lie algebra.*
(4) *$[L, L]$ is a non-trivial irreducible L-module.*

Proof. (1): It is clear that $[L, L]$ is a non-zero ideal of L. Now let I be any non-zero ideal of L. Then the p-ideal $\langle I\rangle_p$ is a non-zero p-ideal of L (cf. [30, Proposition 2.1.3(4)]), and so by hypothesis $\langle I\rangle_p = L$. Hence [30, Proposition 2.1.3(2)] yields $[L, L] = [\langle I\rangle_p, \langle I\rangle_p] = [I, I] \subseteq I$. In particular, $[L, L]$ is a minimal ideal of L, and if I is any minimal ideal of L, then $I = [L, L]$. Consequently, $[L, L]$ is the unique minimal ideal of L.

(2): It follows from the argument used in the proof of (1) that $\langle [L, L]\rangle_p = L$. Accordingly, L is a p-envelope of $[L, L]$. As the center $C(L)$ of L is a p-ideal of L and L is not abelian with no non-zero proper p-ideals, one obtains that $C(L) = 0 \subseteq [L, L]$. Now it is a consequence of [30, Theorem 2.5.8(3)] that L is a minimal p-envelope of $[L, L]$.

(3): Suppose that I is a non-zero ideal of $[L, L]$. By virtue of (2) and [30, Proposition 2.1.3(2)], one has $[L, I] = [\langle [L, L]\rangle_p, I] = [[L, L], I] \subseteq I$, that is, I is also an ideal of L. Then (1) implies that $I = [L, L]$, and thus $[L, L]$ is simple as an ordinary Lie algebra.

Finally, (4) is an immediate consequence of (1) and $C(L) = 0$. □

Theorem 6.4. *Let L be a finite-dimensional restricted Lie algebra over a field of prime characteristic $p > 3$. Then L has maximal 0-PIM if, and only if, L is solvable.*

Proof. One implication is just Theorem 2.2, so that only the converse has to be proved.

Let $\overline{\mathbb{F}}$ denote an algebraic closure of the ground field $\mathbb{F}$ of L, and set $\overline{L} := \overline{\mathbb{F}} \otimes_{\mathbb{F}} L$. As the trivial irreducible L-module $\mathbb{F}$ is absolutely irreducible, it follows that

$$P_{\overline{L}}(\overline{\mathbb{F}}) \cong \overline{\mathbb{F}} \otimes_{\mathbb{F}} P_L(\mathbb{F}) .$$

Hence we may assume that the ground field of L is algebraically closed.

Suppose that there exists a non-solvable restricted Lie algebra L with maximal 0-PIM. Furthermore, we may assume that L has minimal dimension, i.e., every proper p-subalgebra of L is either solvable or does not have maximal 0-PIM.

Suppose now in addition that L has a non-zero proper p-ideal I. According to Lemma 2.1, I and L/I have maximal 0-PIM. By the minimality of L, they both must be solvable, but then also L would be solvable which is a contradiction. So for the remainder of the proof we may assume that L has no non-zero proper p-ideal. Hence it follows from Lemma 6.3 that the derived subalgebra $[L, L]$ of L is a simple Lie algebra and that L is a minimal p-envelope of $[L, L]$.

We first discuss the case that $[L, L]$ has absolute toral rank one. By Engel's theorem there exists an element x of $[L, L]$ such that $\mathrm{ad}x$ is not nilpotent on $[L, L]$. Consider the Jordan-Chevalley decomposition $x = x_s + x_n$ of x in the restricted Lie algebra L, where x_s and x_n denote the semisimple part and the p-nilpotent part of x, respectively (see [30, Theorem 2.3.5]). Then $T := \langle x_s \rangle_p$ is a torus of L of maximal dimension and $[L, L]^T \supseteq \mathbb{F}x \neq 0$. By virtue of Lemma 6.3(4), $[L, L]$ is a non-trivial irreducible restricted L-module. Hence one concludes from Lemma 6.1 that L does not have maximal 0-PIM, which is a contradiction.

Let H be any 2-section of L with respect to a torus T of L of maximal dimension (cf. [26, Definition 1.3.9]). As $T \subseteq H$, one has $\mathrm{MT}(L) = \mathrm{MT}(H)$. Moreover, it follows from Proposition 1.1 that $P_L(\mathbb{F})$ is a direct summand of $\mathrm{Ind}_H^L(P_H(\mathbb{F}), 0)$ and therefore [30, Proposition 5.6.2] yields:

$$\dim_{\mathbb{F}} P_H(\mathbb{F}) \geq \frac{\dim_{\mathbb{F}} P_L(\mathbb{F})}{p^{\dim_{\mathbb{F}} L - \dim_{\mathbb{F}} H}} = p^{\dim_{\mathbb{F}} H - \mathrm{MT}(H)} .$$

Hence, by Proposition 1.2, H has maximal 0-PIM, and the minimality of L implies that either $L = H$ or H is solvable. In the latter case one concludes from [26, Theorem 1.3.10] that L must also be solvable, which is a contradiction. Consequently, $\mathrm{MT}(L) = \mathrm{MT}(H) = 2$, and thus it follows from [26, Lemma 1.2.6(1)] that $[L, L]$ has absolute toral rank two.

We use [28, Theorem, p. 2] (see also [21, Theorem 1.1]) and proceed by a case-by-case analysis. Note that the simple Lie algebras A_2, B_2, G_2, $W(2; \underline{1})$, $S(3; \underline{1})^{(1)}$, $H(4; \underline{1})^{(1)}$, $K(3; \underline{1})$, and $\mathcal{M}(1, 1)$ are restricted, so that $[L, L] = L$. Hence in all these cases one has $[L, L]^T \neq 0$ for any torus T of L, which contradicts Lemma 6.1.

Next, consider $[L, L] \cong W(1; 2)$. Suppose that $[L, L] \cap T = 0$ for some two-dimensional torus T of L. Then it follows from [30, Proposition 4.2.2(2)] and [26, Theorem 7.2.2(1)] that $\dim_{\mathbb{F}}([L, L] + T) > \dim_{\mathbb{F}} L$, which is a contradiction. Hence

$$0 \neq [L, L] \cap T \subseteq [L, L]^T$$

for every two-dimensional torus T of L, which again contradicts Lemma 6.1.

If $[L, L] \cong H(2; (1, 2))^{(2)}$ or $[L, L] \cong H(2; \underline{1}; \Phi(1))$, then it follows from [28, Lemma 10.2.3] and [28, Theorem 10.4.6(1)], respectively, that

$$\dim_{\mathbb{F}}[L, L]^T \geq \dim_{\mathbb{F}}([L, L] \cap T) = 1$$

for every two-dimensional torus T of L, which once more contradicts Lemma 6.1.

Thus we must have $[L, L] \cong H(2; \underline{1}; \Phi(\tau))^{(1)}$. Let T be a two-dimensional torus in L. Moreover, let $(L_{(n)})_{-1 \leq n \leq 2p-4}$ denote the descending restricted filtration of L considered in Section 5. Then $\mathrm{gr}_0(L) \cong \mathfrak{sl}_2(\mathbb{F})$ (see [25, Proposition 1.2(2.b)], and, by [25, Proposition 1.2(2.a)], $\mathrm{gr}_{-1}(L)$ is isomorphic as an $L_{(0)}$-module to the module $V(1)$ inflated from the two-dimensional irreducible $\mathfrak{sl}_2(\mathbb{F})$-module $V(1)$. In particular, since every non-zero torus T_0 of $L_{(0)}$ is one-dimensional and 0 is not a weight of T_0 on $\mathrm{gr}_{-1}(L) \cong V(1)$, it follows that $L_{(0)} \cap T = \{0\}$. Consequently, one concludes by comparison of dimensions that $L = L_{(0)} \oplus T$. According to Theorem 5.3, the L-module $M(2) := \mathrm{Ind}_{L_{(0)}}^{L}(V(2), 0)$ is irreducible. Now, by [30, Theorem 2.3.6(1)], there exists a toral basis $\{t_1, t_2\}$ for T. As $L = L_{(0)} \oplus T$, the subspace $(1 - t_1^{p-1})(1 - t_2^{p-1})M(2)$ of $M(2)$ is non-zero, and it is clear that it is annihilated by T. This assures the existence of a non-trivial irreducible L-module $S := M(2)$ such that $S^T \neq 0$ which again contradicts Lemma 6.1. □

Remark. Note that it follows from [28, Theorem 10.3.2(2)] that

$$[H(2; \underline{1}; \Phi(\tau))^{(1)}]^T = 0$$

for every torus T of maximal dimension in a minimal p-envelope of $H(2; \underline{1}; \Phi(\tau))^{(1)}$. This is the reason why the argument in this case is much more involved than in the other cases.

In the case that $[L, L] \cong W(1; 2)$ one can prove directly that L does not have maximal 0-PIM. Namely, as $p > 3$, [18, Theorem 2.5.9] in conjunction with [26, Theorem 7.2.2(1), Lemma 1.2.6(1), and Theorem 7.6.3(2)] yields that

$$\dim_{\mathbb{F}} P_L(\mathbb{F}) = 2p^{p^2-2} < p^{p^2-1} = p^{\dim_{\mathbb{F}}(L) - \mathrm{MT}(L)}.$$

The preceding argument motivates the following example of a non-solvable restricted Lie algebra in characteristic 2 that has maximal 0-PIM. In fact, the restricted Lie algebra in the example is semisimple as an ordinary Lie algebra (see [12, Proposition 1.4(2)]) and simple as a restricted Lie algebra (see [12, Proposition 1.4(1)]).

Example. Let $W(1; 2) = \mathbb{F}e_{-1} \oplus \mathbb{F}e_0 \oplus \mathbb{F}e_1 \oplus \mathbb{F}e_2$ be a Zassenhaus algebra over an algebraically closed field $\mathbb{F}$ of characteristic 2. (Here we set $e_{-1} := \partial$ and $e_k := x^{(k+1)}\partial$ for $k = 0, 1, 2$, where $\mathcal{O}(1; 2) := \mathbb{F}x^{(0)} \oplus \mathbb{F}x^{(1)} \oplus \mathbb{F}x^{(2)} \oplus \mathbb{F}x^{(3)}$ is a divided power algebra in one variable x over $\mathbb{F}$, $\partial(x^{(0)}) := 0$, and $\partial(x^{(a)}) := x^{(a-1)}$ for $1 \leq a \leq 3$.) Then $W(1; 2)^{(1)} = \mathbb{F}e_{-1} \oplus \mathbb{F}e_0 \oplus \mathbb{F}e_1$ is a simple Lie algebra and $\mathfrak{W} := \mathfrak{W}(1; 2) = \mathbb{F}e_{-2} \oplus \mathbb{F}e_{-1} \oplus \mathbb{F}e_0 \oplus \mathbb{F}e_1 \oplus \mathbb{F}e_2$ (with $e_{-2} := \partial^2$) is a minimal 2-envelope of $W(1; 2)$ as well as of $W(1; 2)^{(1)}$. Note that $\mathfrak{W}$ is the full derivation algebra of $W(1; 2)^{(1)}$ (see [27, Proposition 3.3(1)]), and as such $\mathfrak{W}$ is a restricted Lie algebra. The elements $t_{\pm} := e_0 + e_{\pm 1} + e_{\pm 2}$ are toral, and $\mathfrak{T} := \mathbb{F}t_+ \oplus \mathbb{F}t_-$ is a two-dimensional torus of $\mathfrak{W}$. It follows from [26, Theorem 7.6.3(2) and Lemma 1.2.6(1)] that $\mathfrak{T}$ is a torus of maximal dimension.

Set $\mathfrak{B} := \mathbb{F}e_0 \oplus \mathbb{F}e_1 \oplus \mathbb{F}e_2$. Since $\mathfrak{B}$ is a 2-subalgebra of $\mathfrak{W}$ with a one-dimensional torus $\mathbb{F}e_0$ and a two-dimensional 2-radical $\mathbb{F}e_1 \oplus \mathbb{F}e_2$ (see [30, p. 68]), $\mathfrak{B}$ has two isomorphism classes of irreducible restricted modules, which can be represented by $F_\lambda := \mathbb{F}1_\lambda$ ($\lambda = 0, 1$), where $e_0 \cdot 1_\lambda := \lambda 1_\lambda$ and $e_k \cdot 1_\lambda := 0$ for $k = 1, 2$ (cf. [31, Lemma 2.4] or [30, Lemma 5.8.6(2)]).

Consider the restricted baby Verma modules $Z(\lambda) := \mathrm{Ind}_{\mathfrak{B}}^{\mathfrak{W}}(F_\lambda, 0)$ for weights $\lambda = 0, 1$, and let S denote any irreducible restricted $\mathfrak{W}$-module. Then it follows from Frobenius reciprocity that $\operatorname{Hom}_{\mathfrak{W}}(Z(\lambda), S) \cong \operatorname{Hom}_{\mathfrak{B}}(F_\lambda, S_{|\mathfrak{B}}) \neq 0$ for some $\lambda = 0$ or $\lambda = 1$, and therefore S is a homomorphic image of $Z(\lambda)$ for $\lambda = 0$ or $\lambda = 1$. On the other hand, one concludes from [12, Proposition 4.10 and Proposition 4.7(1) and (2)] that $Z(0) \cong W(1;2)$ and $Z(1) \cong W(1;2)^*$. According to Lemma 6.3(1), $W(1;2)^{(1)} = [\mathfrak{W}, \mathfrak{W}]$ is the unique minimal $\mathfrak{W}$-submodule of $W(1;2)$. Because of $\dim_{\mathbb{F}} W(1;2)/W(1;2)^{(1)} = 1$, $W(1;2)^{(1)} = [\mathfrak{W}, \mathfrak{W}]$ is also the unique maximal $\mathfrak{W}$-submodule of $W(1;2)$. It follows that $Z(\lambda)$ has a unique maximal submodule, and therefore $Z(\lambda)$ has a unique irreducible factor module. Hence $\mathfrak{W}$ has at most two isomorphism classes of irreducible restricted modules.

By [12, Proposition 1.4(1)] and Lemma 6.3(4), $W(1;2)^{(1)} = [\mathfrak{W}, \mathfrak{W}]$ is a non-trivial irreducible $\mathfrak{W}$-module. So $\mathfrak{W}$ has exactly two isomorphism classes of irreducible restricted modules, and up to isomorphism $W(1;2)^{(1)}$ is the only non-trivial irreducible restricted $\mathfrak{W}$-module. Finally, it follows from $[W(1;2)^{(1)}]^{\mathfrak{T}} = 0$ and Lemma 6.1 that $\mathfrak{W}$ has maximal 0-PIM, but $\mathfrak{W}$ is not solvable as it contains the simple subalgebra $W(1;2)^{(1)}$.

In particular, by the argument used in the proof of Theorem 3.1 one obtains that $|\operatorname{Irr}(\mathfrak{W}, \chi)| \leq 4$ for every linear form χ on $\mathfrak{W}$. However, one would even expect that $|\operatorname{Irr}(\mathfrak{W}, \chi)| \leq 2$ as in the restricted case. Moreover, it is not difficult to show that the projective cover of the non-trivial irreducible restricted $\mathfrak{W}$-module $W(1;2)^{(1)} = [\mathfrak{W}, \mathfrak{W}]$ has the same dimension as the projective cover of the one-dimensional trivial $\mathfrak{W}$-module. We will leave the details to the interested reader and refer to [16, Theorem 3.5] for the more general case of the minimal 2-envelope of $W(1;n)$.

Acknowledgements. We are very grateful to Sasha Premet and to the referees for their comments that helped to improve the exposition of the paper. In particular, we wish to thank one of the referees for suggesting an alternative approach to Theorem 2.2 via Lemma 6.2 and for providing a direct proof of the latter. The first and the second author would like to thank the Dipartimento di Matematica e Applicazioni at the Università degli Studi di Milano-Bicocca for the hospitality during their visit in May 2013. The third author would like to thank the Department of Mathematics and Statistics at the University of South Alabama for the hospitality during his visit in October 2013.

References

[1] R. E. Block and R. L. Wilson, *Classification of the restricted simple Lie algebras*, J. Algebra **114** (1988), no. 1, 115–259.

[2] R. Farnsteiner, *Extension functors of modular Lie algebras*, Math. Ann. **288** (1990), no. 4, 713–730.

[3] R. Farnsteiner, *Representations of Lie algebras with triangular decomposition*, in: Proceedings of the 2nd Gauss Symposium, Conference A: Mathematics and Theoretical Physics, Munich, 1993 (M. Behara, R. Fritsch, and R. G. Lintz, eds.); Sympos. Gaussiana, Walter de Gruyter, Berlin/New York, 1995, 275–286.

[4] R. Farnsteiner, *Representations of blocks associated to induced modules of restricted Lie algebras*, Math. Nachr. **179** (1996), 57–88.

[5] R. Farnsteiner and H. Strade, *Shapiro's lemma and its consequences in the cohomology theory of modular Lie algebras*, Math. Z. **206** (1991), no. 1, 153–168.

[6] J. Feldvoss, *Homologische Aspekte der Darstellungstheorie modularer Lie-Algebren*, Doctoral Dissertation, Universität Hamburg, 1989.

[7] J. Feldvoss, *On the cohomology of restricted Lie algebras*, Comm. Algebra **19** (1991), no. 10, 2865–2906.

[8] J. Feldvoss, *Blocks and projective modules for reduced universal enveloping algebras of a nilpotent restricted Lie algebra*, Arch. Math. (Basel) **65** (1995), no. 6, 495–500.

[9] J. Feldvoss, *Homological topics in the representation theory of restricted Lie algebras*, in: Lie algebras and their representations (S.-J. Kang, M.-H. Kim, and I. Lee, eds.); Contemp. Math., vol. 194, Amer. Math. Soc., Providence, RI, 1996, 69–119.

[10] J. Feldvoss, *Projective modules and blocks of supersolvable restricted Lie algebras*, J. Algebra **222** (1999), no. 1, 284–300.

[11] J. Feldvoss, *On the number of simple modules of a supersolvable restricted Lie algebra*, J. Algebra **230** (2000), no. 2, 319–333.

[12] J. Feldvoss, *On the cohomology of modular Lie algebras*, in: Lie algebras, vertex operator algebras and their applications, Raleigh, NC, 2005 (Y.-Z. Huang and K. C. Misra, eds.); Contemp. Math., vol. 442, Amer. Math. Soc., Providence, RI, 2007, 89–113.

[13] R. R. Holmes, *Simple restricted modules for the restricted Hamiltonian algebra*, J. Algebra **199** (1998), no. 1, 229–261.

[14] J. E. Humphreys, *Modular representations of simple Lie algebras*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 2, 105–122.

[15] B. Huppert and N. Blackburn, *Finite groups II*, Grundlehren der Mathematischen Wissenschaften, vol. 242, Springer-Verlag, Berlin/Heidelberg/New York, 1982.

[16] B. Lancelotti and Th. Weigel, *The p.i.m.s for the restricted Zassenhaus algebras in characteristic* 2, Arch. Math. (Basel) **104** (2015), no. 4, 333–340.

[17] G. Malle and Th. Weigel, *Finite groups with minimal* 1*-PIM*, Manuscripta Math. **126** (2008), no. 3, 315–332.

[18] D. K. Nakano, *Projective modules over Lie algebras of Cartan type*, Mem. Amer. Math. Soc. **98** (1992), no. 470.

[19] A. Premet and H. Strade, *Simple Lie algebras of small characteristic I: Sandwich elements*, J. Algebra **189** (1997), no. 2, 419–480.

[20] A. Premet and H. Strade, *Simple Lie algebras of small characteristic II: Exceptional roots*, J. Algebra **216** (1999), no. 1, 190–301.

[21] A. Premet and H. Strade, *Simple Lie algebras of small characteristic III: The toral rank* 2 *case*, J. Algebra **242** (2001), no. 1, 236–337.

[22] A. Premet and H. Strade, *Classification of finite dimensional simple Lie algebras in prime characteristics*, in: Representations of algebraic groups, quantum groups, and Lie algebras, Snowbird, Utah, 2004 (G. Benkart, J. C. Jantzen, Z. Lin, D. K. Nakano, and B. J. Parshall, eds.); Contemp. Math., vol. 413, Amer. Math. Soc., Providence, RI, 2006.

[23] G.-Y. Shen, *Graded modules of graded Lie algebras of Cartan type III: Irreducible modules*, Chinese Ann. Math. Ser. B **9** (1988), no. 4, 404–417.

[24] H. Strade, *Darstellungen auflösbarer Lie-p-Algebren*, Math. Ann. **232** (1978), no. 1, 15–32.

[25] H. Strade, *Representations of the* $(p^2 - 1)$*-dimensional Lie algebras of R. E. Block*, Canad. J. Math. **43** (1991), no. 3, 580–616.

[26] H. Strade, *Simple Lie algebras over fields of positive characteristic I: Structure theory*, de Gruyter Expositions in Mathematics, vol. 38, Walter de Gruyter, Berlin/New York, 2004.

[27] H. Strade, *Lie algebras of small dimension*, in: Lie algebras, vertex operator algebras and their applications, Raleigh, NC, 2005 (Y.-Z. Huang and K. C. Misra, eds.); Contemp. Math., vol. 442, Amer. Math. Soc., Providence, RI, 2007, pp. 233–265.

[28] H. Strade, *Simple Lie algebras over fields of positive characteristic II: Classifying the absolute toral rank two case*, de Gruyter Expositions in Mathematics, vol. 42, Walter de Gruyter, Berlin/New York, 2009.

[29] H. Strade, *Simple Lie algebras over fields of positive characteristic III: Completion of the classification*, de Gruyter Expositions in Mathematics, vol. 57, Walter de Gruyter, Berlin/New York, 2013.

[30] H. Strade and R. Farnsteiner, *Modular Lie algebras and their representations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 116, Marcel Dekker, New York/Basel, 1988.

[31] B. Yu. Veisfeiler and V. G. Kats, *Irreducible representations of Lie p-algebras*, Funct. Anal. Appl. **5** (1971), no. 2, 111–117.

[32] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[33] W. Willems, *On the projectives of a group algebra*, Math. Z. **171** (1980), no. 2, 163–174.

[34] D. J. Winter, *On the toral structure of Lie p-algebras*, Acta Math. **123** (1969), 69–81.