

SPLIT ABELIAN CHIEF FACTORS AND FIRST DEGREE COHOMOLOGY FOR LIE ALGEBRAS

JÖRG FELDVOSS, SALVATORE SICILIANO, AND THOMAS WEIGEL

Dedicated to Helmut Strade on the occasion of his seventieth birthday

ABSTRACT. In this paper we investigate the relation between the multiplicities of split abelian chief factors of finite-dimensional Lie algebras and first degree cohomology. In particular, we obtain a characterization of modular solvable Lie algebras in terms of the vanishing of first degree cohomology or in terms of the multiplicities of split abelian chief factors. The analogues of these results are well known in the modular representation theory of finite groups. An important tool in the proof of these results is a refinement of a non-vanishing theorem of Seligman for the first degree cohomology of non-solvable finite-dimensional Lie algebras in prime characteristic. As an application we derive several results in the representation theory of restricted Lie algebras related to the principal block and the projective cover of the trivial irreducible module of a finite-dimensional restricted Lie algebra. In particular, we obtain a characterization of solvable restricted Lie algebras in terms of the second Loewy layer of the projective cover of the trivial irreducible module.

1. INTRODUCTION

W. Gaschütz proved the “only if”-part of the following cohomological vanishing theorem for finite p -solvable groups (see [22, Lemma 1]), and the converse is due to U. Stambach [22, Theorem A]. Here and in the following p is an arbitrary prime number, and G is a finite group whose order is divisible by p . Moreover, let $\mathbb{F}_p[G]$ denote the group algebra of G over the field \mathbb{F}_p with p elements, and let $C_G(M) := \{g \in G \mid g \cdot m = m \text{ for every } m \in M\}$ denote the *centralizer* of a (unital left) $\mathbb{F}_p[G]$ -module M in G .

Theorem 1.1. *A finite group G is p -solvable if, and only if, $H^1(G/C_G(S), S) = 0$ for every irreducible $\mathbb{F}_p[G]$ -module S .*

Let S be an irreducible $\mathbb{F}_p[G]$ -module. Then $[G : S]_{\text{split}}$ denotes the number of p -elementary abelian chief factors G_j/G_{j-1} ($1 \leq j \leq n$) of a given chief series $\{1\} = G_0 \subset G_1 \subset \dots \subset G_n = G$ that are isomorphic to S as $\mathbb{F}_p[G]$ -modules and for which the exact sequence $\{1\} \rightarrow G_j/G_{j-1} \rightarrow G/G_{j-1} \rightarrow G/G_j \rightarrow \{1\}$ splits in the category of groups. According to the main result of [23], $[G : S]_{\text{split}}$ is independent of the choice of the chief series of G .

Date: January 21, 2013.

2010 Mathematics Subject Classification. 17B05, 17B30, 17B50, 17B55, 17B56.

Key words and phrases. Solvable Lie algebra, irreducible module, split abelian chief factor, cohomology, restricted cohomology, Loewy layer, projective indecomposable module, principal block.

W. Gaschütz also proved the “only if”-part of the following result on split (or complementable) p -chief factors of finite p -solvable groups (see [17, Theorem VII.15.5]). The converse of Gaschütz’ theorem is due to U. Stambach [23, Corollary 1], but in an equivalent form it was already proved earlier by W. Willems [25, Theorem 3.9].

Theorem 1.2. *A finite group G is p -solvable if, and only if, $\dim_{\mathbb{F}_p} H^1(G, S) = \dim_{\mathbb{F}_p} \text{End}_{\mathbb{F}_p[G]}(S) \cdot [G : S]_{\text{split}}$ holds for every irreducible $\mathbb{F}_p[G]$ -module S .*

The main goal of this paper is to investigate whether analogues of Theorem 1.1 and Theorem 1.2 hold in the context of ordinary Lie algebras. Some time ago in an unpublished manuscript the first author of this paper has obtained analogues of these results for restricted Lie algebras (see [11, Remark after Theorem 7]). The crucial point in the argument is an analogue of Theorem 1.1 for ordinary Lie algebras (see [12, Proposition 1]). The proof in [12] was a modification of the proof of a similar characterization of supersolvable Lie algebras due to D. W. Barnes (see [2, Theorem 4]). It turns out that there is a gap in both these proofs which will be fixed in the present paper. This is achieved by applying Shapiro’s lemma for truncated coinduced modules (see [5, Theorem in §5 and Corollary 1 in §3] or [8, Theorem 2.1 and Theorem 1.4]) in order to establish a refinement of a non-vanishing theorem of G. Seligman for the first degree cohomology of a non-solvable finite-dimensional Lie algebra in prime characteristic (see [21, p. 102]). In this respect our approach is similar to Stambach’s proof of the “if”-part of Theorem 1.1 (see the proof of [22, Lemma 2]).

We begin with a Lie-theoretic analogue of the main result of [23] (see Theorem 2.1). This result is already contained in the first author’s unpublished manuscript (see [11, Lemma 5]), but the proof there implicitly uses that the multiplicity of split abelian chief factors isomorphic to a given irreducible module is independent of the chief series. The proof given here follows the argument used in the proof of [11, Lemma 5], but also deals with a case not considered in [11]. As a consequence of Theorem 2.1 and Barnes’ cohomological vanishing theorem for solvable Lie algebras (see [1, Theorem 2]), one obtains the Lie-theoretic analogue of Gaschütz’ theorem on split p -chief factors (see Theorem 2.3). In the third section we show that for fields of characteristic zero the analogues of the conditions in Theorem 1.1 and Theorem 1.2 are always satisfied. Thus, as in the group case, only modular solvable Lie algebras can be characterized by these properties. This will be the main goal of the fourth section (see Theorem 4.3). In the final section we apply the results obtained in Sections 2 and 4 to restricted Lie algebras. The equivalence (i) \iff (viii) in Theorem 5.5 is an analogue of Willems’ module-theoretic characterization of p -solvable groups (see [25, Theorem 3.9]) for restricted Lie algebras. As a by-product we establish several results on the second Loewy layer of the projective cover of the trivial irreducible module. Most of the results in Section 5 were already contained in [11, Section 4], but have never been published before.

2. SPLIT ABELIAN CHIEF FACTORS AND FIRST DEGREE COHOMOLOGY

In analogy to group theory we define a *chief series* for a finite-dimensional Lie algebra L to be an ascending chain $0 = L_0 \subset L_1 \subset \dots \subset L_n = L$ of ideals in L such that L_j/L_{j-1} is a minimal (non-zero) ideal of L/L_{j-1} for every integer j with $1 \leq j \leq n$. Any L_j/L_{j-1} is then called a *chief factor* of L , and we say that L_j/L_{j-1}

is an *abelian chief factor* if it is an abelian Lie algebra. For a finite-dimensional irreducible L -module S and a given chief series $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ of L we denote by $[L : S]_{\text{split}}$ the number of abelian chief factors L_j/L_{j-1} that are isomorphic to S as an L -module and for which the exact sequence $0 \rightarrow L_j/L_{j-1} \rightarrow L/L_{j-1} \rightarrow L/L_j \rightarrow 0$ splits in the category of Lie algebras. Observe that any composition series of the adjoint L -module is also a chief series of L and vice versa. But note that split chief factors do not necessarily split in the category of L -modules, e.g., this happens for the one-dimensional ideal of the non-abelian two-dimensional Lie algebra. However, only abelian chief factors will be relevant for our purpose. (Note that every chief factor of a solvable Lie algebra is abelian.) As we will show that $[L : S]_{\text{split}}$ is independent of the choice of the chief series, we will not indicate the chief series in the notation.

Our first result is completely analogous to the main result of [23] and uses [2, Lemma 2]:

Theorem 2.1. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of arbitrary characteristic, and let S be a finite-dimensional irreducible L -module with centralizer algebra $\mathbb{D} := \text{End}_L(S)$. Then*

$$(1) \quad [L : S]_{\text{split}} = \dim_{\mathbb{D}} H^1(L, S) - \dim_{\mathbb{D}} H^1(L/\text{Ann}_L(S), S)$$

holds. In particular, $[L : S]_{\text{split}}$ is independent of the choice of the chief series of L .

Proof. We proceed by induction on the dimension of L . If L is one-dimensional, then the assertion is easy to check. Thus we may assume that the dimension of L is greater than one, and that the claim holds for all Lie algebras of dimension less than $\dim_{\mathbb{F}} L$. Let $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ be a chief series of L . For the remainder of the proof the multiplicity $[L : S]_{\text{split}}$ always refers to this fixed chief series.

If $\text{Ann}_L(S) = 0$, then the right-hand side of (1) is zero. But as abelian chief factors have non-zero annihilators, the left-hand side also vanishes and the assertion holds. So we may assume that $\text{Ann}_L(S) \neq 0$.

Firstly, we assume that L_1 is contained in $\text{Ann}_L(S)$. Then the five-term exact sequence for Lie algebra cohomology (see [16, Theorem 6]) specializes to

$$(2) \quad 0 \rightarrow H^1(L/L_1, S) \rightarrow H^1(L, S) \rightarrow \text{Hom}_L(L_1/[L_1, L_1], S) \rightarrow H^2(L/L_1, S).$$

Since S is also an irreducible L/L_1 -module, one obtains by induction that

$$(3) \quad [L/L_1 : S]_{\text{split}} = \dim_{\mathbb{D}} H^1(L/L_1, S) - \dim_{\mathbb{D}} H^1(L/\text{Ann}_L(S), S).$$

As L_1 is a minimal ideal of L , L_1 is either perfect or abelian. In the former case, the third term in (2) vanishes, and therefore $H^1(L/L_1, S) \cong H^1(L, S)$. Since L_1 is not abelian, one has $[L : S]_{\text{split}} = [L/L_1 : S]_{\text{split}}$. Hence (1) holds in this case.

If L_1 is abelian, one has $\text{Hom}_L(L_1/[L_1, L_1], S) = \text{Hom}_L(L_1, S)$. If L_1 and S are not isomorphic as L -modules, then $\text{Hom}_L(L_1, S)$ vanishes, and the assertion follows as before.

For $L_1 \cong S$ one has to distinguish two cases: The abelian chief factor L_1 is split, or not split. In case that L_1 is split, one has

$$(4) \quad \begin{aligned} [L : S]_{\text{split}} &= [L/L_1 : S]_{\text{split}} + 1 \\ &= \dim_{\mathbb{D}} H^1(L/L_1, S) - \dim_{\mathbb{D}} H^1(L/\text{Ann}_L(S), S) + 1, \end{aligned}$$

and [2, Lemma 2(a)] shows that the transgression $\text{Hom}_L(L_1, S) \rightarrow H^2(L/L_1, S)$ is zero. Thus the exactness of (2) implies that the restriction $H^1(L, S) \rightarrow \text{Hom}_L(L_1, S)$ is surjective, and therefore

$$(5) \quad \begin{aligned} \dim_{\mathbb{D}} H^1(L, S) &= \dim_{\mathbb{D}} H^1(L/L_1, S) + \dim_{\mathbb{D}} \text{Hom}_L(L_1, S) \\ &= \dim_{\mathbb{D}} H^1(L/L_1, S) + 1. \end{aligned}$$

Thus (4) and (5) yield the assertion. Suppose that L_1 is not split. In this case the transgression $\text{Hom}_L(L_1, S) \rightarrow H^2(L/L_1, S)$ is injective (see [2, Lemma 2(b)]). According to (2), the inflation $H^1(L/L_1, S) \rightarrow H^1(L, S)$ is bijective. Then one has $[L : S]_{\text{split}} = [L/L_1 : S]_{\text{split}}$. Hence the claim follows from (3).

Finally, assume that L_1 is not contained in $\text{Ann}_L(S)$, i.e., $L_1 \cap \text{Ann}_L(S) = 0$ and $S^{L_1} = 0$. Suppose that L_j/L_{j-1} is abelian and $L_j/L_{j-1} \cong S$ as L -modules for some integer j with $1 \leq j \leq n$. Then L_j – and therefore L_1 – would be contained in $\text{Ann}_L(S)$, a contradiction. Consequently, $[L : S]_{\text{split}} = 0$. As $S^{L_1} = 0$, one concludes from the five-term exact sequence

$$0 \longrightarrow H^1(L/L_1, S^{L_1}) \longrightarrow H^1(L, S) \longrightarrow H^1(L_1, S)^L \longrightarrow H^2(L/L_1, S^{L_1})$$

that the vertical mappings in the commutative diagram

$$\begin{array}{ccc} H^1(L/\text{Ann}_L(S), S) & \xrightarrow{\alpha} & H^1(L, S) \\ \downarrow & & \downarrow \\ H^1(L_1 + \text{Ann}_L(S)/\text{Ann}_L(S), S)^L & \xrightarrow{\beta} & H^1(L_1, S)^L \end{array}$$

are isomorphisms. Hence, because β is an isomorphism, α is an isomorphism as well. This shows that in this case the right-hand side of (1) is also zero.

Since the right-hand side of (1) does not depend on the choice of the chief series, the left-hand side does not either. This completes the proof of the theorem. \square

In the extreme case $\text{Ann}_L(S) = L$, Theorem 2.1 has the following implication:

Corollary 2.2. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of arbitrary characteristic. Then the trivial irreducible L -module occurs with multiplicity $\dim_{\mathbb{F}} L/[L, L]$ as a split abelian chief factor of L .*

The analogue of Gaschütz' theorem on split p -chief factors (see the ‘‘only if’’-part of Theorem 1.2 in the introduction) for solvable Lie algebras is another immediate consequence of Theorem 2.1 in conjunction with [1, Theorem 2] and generalizes [2, Theorem 1].

Theorem 2.3. *Let L be a finite-dimensional solvable Lie algebra over a field \mathbb{F} of arbitrary characteristic. Then*

$$\dim_{\mathbb{F}} H^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$$

holds for every finite-dimensional irreducible L -module S .

The same is also true for any finite-dimensional Lie algebra in characteristic zero (see the remark after [11, Lemma 5]) as we will prove in the next section.

3. LIE ALGEBRAS IN CHARACTERISTIC ZERO

We begin by proving that in characteristic zero the cohomology of finite-dimensional faithful irreducible modules always vanishes (see the remark after [12, Proposition 1]):

Theorem 3.1. *Let $L \neq 0$ be a finite-dimensional Lie algebra over a field of characteristic zero, and let S be a finite-dimensional faithful irreducible L -module. Then $H^n(L, S) = 0$ for every non-negative integer n .*

Proof. Since S is faithful and irreducible, we have that $S^L = 0$. If L is semisimple, then the assertion is an immediate consequence of Whitehead's cohomological vanishing theorem (see [19, Theorem 14, p. 96]). Otherwise L has a non-zero abelian ideal I .

As S^I is an L -submodule of the faithful irreducible L -module S , we obtain that $S^I = 0$. Then it follows from [1, Theorem 1] that $H^n(I, S) = 0$ for every positive integer n . Finally, from the Hochschild-Serre spectral sequence (see [16, Theorem 6]) one concludes that $H^n(L, S) \cong H^n(L/I, S^I) = 0$ for every positive integer n . \square

In particular, we have the following characteristic zero version of Barnes' cohomological vanishing theorem [1, Theorem 2]:

Corollary 3.2. *Let L be a finite-dimensional Lie algebra over a field of characteristic zero, and let S be a finite-dimensional irreducible L -module. Then*

$$H^n(L/\text{Ann}_L(S), S) = 0$$

for every positive integer n .

As a consequence of Theorem 2.1 and Corollary 3.2, we obtain the following result on split abelian chief factors and first degree cohomology in characteristic zero which generalizes [2, Theorem 2]:

Theorem 3.3. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of characteristic zero. Then*

$$\dim_{\mathbb{F}} H^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$$

holds for every finite-dimensional irreducible L -module S .

4. LIE ALGEBRAS IN PRIME CHARACTERISTIC

The next result will be essential in proving Theorem 4.2 and thereby is correcting the gap in the proof of [12, Proposition 1]. Let $U(L)$ denote the universal enveloping algebra of a Lie algebra L over a field of prime characteristic p and recall that $\mathcal{O}(L, I)$ is the unital associative subalgebra of $U(L)$ generated by $I \cup \{z_1, \dots, z_k\}$, where I is an ideal of L and the elements $z_i := e_i^{p^{m_i}} + v_i$ belong to the center of $U(L)$, with $\{e_1, \dots, e_k\}$ a cobasis of I in L and $v_i \in U(L)_{(p^{m_i}-1)}$ (see [8, p. 154]).

Proposition 4.1. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic, let I be an ideal in L of codimension one, and let S be an irreducible I -module. Then every composition factor of the restriction of the truncated co-induced module $\text{Hom}_{\mathcal{O}(L, I)}(U(L), S)$ to I is isomorphic to S .*

Proof. Since I has codimension one in L , there exists $t \in L$ such that $L = \mathbb{F}t \oplus I$. Consider first the induced module $M := U(L) \otimes_{U(I)} S^*$, where S^* denotes the linear dual of S which is also an irreducible I -module. For any non-negative integer n put

$$\mathcal{F}^n(M) := \sum_{0 \leq \nu \leq n} t^\nu \otimes S^*.$$

By virtue of the Cartan-Weyl formula (see [24, Proposition 1.1.3(4)]), $\mathcal{F}^n(M)$ is an I -submodule of M for every n , and $(\mathcal{F}^n(M))_{n \geq 0}$ is an exhaustive increasing filtration of M . Set $\text{gr}^n(M) := \mathcal{F}^n(M)/\mathcal{F}^{n-1}(M)$ for any non-negative integer n . (Here $\mathcal{F}^{-1}(M) := 0$.) Then left multiplication by t induces an isomorphism $\text{gr}^n(M) \rightarrow \text{gr}^{n+1}(M)$ of I -modules. In particular, $\text{gr}^n(M) \cong S^*$ for every non-negative integer n . Hence $(\mathcal{F}^n(M))_{n \geq 0}$ is a composition series of M considered as an I -module.

Let $\bar{M} := U(L) \otimes_{\mathcal{O}(L,I)} S^*$ denote the truncated induced L -module of S^* . It follows from [6, p. 35] (see also [14, Theorem 4.4(1)]) that \bar{M} is a factor module of M , i.e., there exists an L -module epimorphism $\pi : M \rightarrow \bar{M}$. Since \bar{M} is finite-dimensional, there exists a positive integer n such that π is mapping $\mathcal{F}^n(M)$ onto \bar{M} . Thus every composition factor of the restriction of \bar{M} to I is isomorphic to S^* . As $\text{Hom}_{\mathcal{O}(L,I)}(U(L), S)$ is isomorphic to \bar{M}^* (see the proof of [14, Theorem 4.1]), every composition factor of the restriction of $\text{Hom}_{\mathcal{O}(L,I)}(U(L), S)$ to I is isomorphic to S . \square

The following is a refinement of a cohomological non-vanishing theorem of G. Seligman [21, p. 102] for Lie algebras that are not solvable. Note that Seligman's result is a consequence of a result of N. Jacobson [18, Theorem 2] (see also [19, Theorem 2, p. 205]) stating that every finite-dimensional Lie algebra over a field of prime characteristic has finite-dimensional modules that are not completely reducible.

Theorem 4.2. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic. If L is not solvable, then there exists an irreducible L -module S such that $H^1(L/\text{Ann}_L(S), S) \neq 0$.*

Proof. Suppose that the assertion is false. Then there exists a non-solvable Lie algebra L of minimal dimension with the property that $H^1(L/\text{Ann}_L(S), S) = 0$ for every irreducible L -module S .

We first show that every proper factor algebra of L is solvable. Suppose on the contrary that $\bar{L} := L/K$ is not solvable for some non-zero ideal K of L . Since \bar{L} has smaller dimension than L , there exists an irreducible \bar{L} -module \bar{S} satisfying $H^1(\bar{L}/\text{Ann}_{\bar{L}}(\bar{S}), \bar{S}) \neq 0$. Let S denote the L -module which is equal to \bar{S} as a vector space and whose L -action is induced by the \bar{L} -action on \bar{S} . Hence, by construction, $\text{Ann}_{\bar{L}}(\bar{S}) = \text{Ann}_L(S)/K$, and therefore $H^1(L/\text{Ann}_L(S), S) \cong H^1(\bar{L}/\text{Ann}_{\bar{L}}(\bar{S}), \bar{S}) \neq 0$, a contradiction.

Next, we prove that L has a unique minimal ideal. Suppose on the contrary that there exist two different minimal ideals J_1 and J_2 of L . The minimality of J_1 and J_2 implies that $J_1 \cap J_2 = 0$, and thus L can be embedded into $L/J_1 \times L/J_2$, where the latter is solvable. This is a contradiction, and therefore L has a unique minimal ideal which we denote by $\text{Soc}(L)$.

Since $\text{Soc}(L)$ is non-zero, $L/\text{Soc}(L)$ is solvable. Moreover, $L/\text{Soc}(L) \neq 0$. Suppose on the contrary that $L = \text{Soc}(L)$. As $\text{Soc}(L)$ is the unique minimal ideal of L , it is contained in every non-zero ideal of L , and so we obtain that L is simple.

According to a result of Seligman [21, p. 102] and the long exact sequence in cohomology, there exists an irreducible L -module S satisfying $H^1(L, S) \neq 0$. As L is perfect, one has $H^1(L, \mathbb{F}) = 0$. Hence S is non-trivial, and by the simplicity of L , we obtain that $\text{Ann}_L(S) = 0$. Consequently, L cannot be a counterexample to the assertion of the theorem which again is a contradiction.

Now we prove that L has an ideal I of codimension one. In particular, I is not solvable. Since $L/\text{Soc}(L)$ is a non-zero solvable Lie algebra, one concludes that $L/[L, L] \neq 0$. Choose a subspace I in L of codimension one that contains $[L, L]$. Then I is an ideal in L of codimension one.

As I is not solvable of codimension one in L , there exists an irreducible I -module S satisfying $H^1(I/\text{Ann}_I(S), S) \neq 0$. According to [1, Theorem 2], $I/\text{Ann}_I(S)$ is not solvable. Since $I/\text{Soc}(L)$ is a solvable Lie algebra, $\text{Soc}(L) \not\subseteq \text{Ann}_I(S)$. As a consequence, $\text{Ann}_I(S) = 0$, and thus also $S^I = 0$ as well as $H^1(I, S) \neq 0$.

It follows from [5, Theorem in §5 and Corollary 1 in §3] or [8, Theorem 2.1 and Theorem 1.4] that

$$H^1(L, \text{Hom}_{\mathcal{O}(L, I)}(U(L), S)) \cong H^1(I, S) \oplus (L/I) \otimes S^I = H^1(I, S) \neq 0.$$

Hence by the long exact sequence in cohomology there exists a composition factor X of the L -module $\text{Hom}_{\mathcal{O}(L, I)}(U(L), S)$ such that $H^1(L, X) \neq 0$.

According to Proposition 4.1, every composition factor of the restriction of $\text{Hom}_{\mathcal{O}(L, I)}(U(L), S)$ to I is isomorphic to S . Since X is a composition factor of $\text{Hom}_{\mathcal{O}(L, I)}(U(L), S)$, every composition factor of the restriction of X to I is also isomorphic to S . In particular, the socle of X is isomorphic to a direct sum of copies of S . Consequently, $\text{Ann}_I(X) \subseteq \text{Ann}_I(S) = 0$. Suppose that $\text{Ann}_L(X) \neq 0$. Then $0 \neq \text{Soc}(L) \subseteq \text{Ann}_L(X) \cap I = \text{Ann}_I(X)$ which is a contradiction. Hence $\text{Ann}_L(X) = 0$. But this shows that X is a faithful irreducible L -module such that $H^1(L, X) \neq 0$ contradicting the choice of L . \square

We are ready to prove the following characterization of solvable Lie algebras over fields of prime characteristic which is the Lie-theoretic analogue of [22, Theorem A] and [23, Corollary 1] (see Theorem 1.1 and Theorem 1.2 in the introduction). The equivalence of (i) and (ii) in Theorem 4.3 also corrects the gap in the proof of [12, Proposition 1].

Theorem 4.3. *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic. Then the following statements are equivalent:*

- (i) L is solvable.
- (ii) $H^1(L/\text{Ann}_L(S), S) = 0$ for every irreducible L -module S .
- (iii) $\dim_{\mathbb{F}} H^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$ holds for every irreducible L -module S .

Proof. The implication (i) \implies (ii) is just [1, Theorem 2], and the equivalence of (ii) and (iii) follows from Theorem 2.1. Finally, the remaining implication (ii) \implies (i) is just the contraposition of the statement of Theorem 4.2. \square

Remark. Both the proofs of [2, Theorem 4] and [12, Proposition 1] use that for a finite-dimensional Lie algebra L with unique minimal ideal I the vanishing of $H^1(I, I)^L$ implies that $[L, L] \subseteq I$. This is not true as the following example shows: For $L := \text{Der}(\mathfrak{psl}_3(\mathbb{F}))$ and $I := \mathfrak{psl}_3(\mathbb{F})$ over any field \mathbb{F} of characteristic three one has that $L/I \cong I$, and I is the unique minimal ideal of L . Moreover, $H^1(I, I)^L = 0$, but $[L, L] = L$.

The statement of [12, Proposition 1] is the equivalence of (i) and (ii) in Theorem 4.3. A proof of [2, Theorem 4] can be obtained along the lines of the argument in [12], when [12, Proposition 1] is replaced by Theorem 4.3.

Corollary 3.2 and Theorem 3.3 show that (ii) and (iii) in Theorem 4.3 are always true in characteristic zero. So these conditions do not characterize solvable Lie algebras in this case.

5. RESTRICTED LIE ALGEBRAS

Let A be a finite-dimensional (unital) associative algebra with Jacobson radical $\text{Jac}(A)$, and let M be a (unital left) A -module. Then the descending filtration

$$M \supset \text{Jac}(A)M \supset \text{Jac}(A)^2M \supset \text{Jac}(A)^3M \supset \cdots \supset \text{Jac}(A)^\ell M \supset \text{Jac}(A)^{\ell+1}M = 0$$

is called the *Loewy series* of M and the factor module $\text{Jac}(A)^{n-1}M / \text{Jac}(A)^nM$ is called the n^{th} *Loewy layer* of M (see [3, Definition 1.2.1] or [17, Definition VII.10.10a]).

Recall that a projective module $P_A(M)$ is a *projective cover* of M , if there exists an A -module epimorphism π_M from $P_A(M)$ onto M such that the kernel of π_M is contained in the radical $\text{Jac}(A)P_A(M)$ of $P_A(M)$. If projective covers exist, then they are unique up to isomorphism. It is well known that projective covers of finite-dimensional modules over finite-dimensional associative algebras exist and are again finite-dimensional. Moreover, every projective indecomposable A -module is isomorphic to the projective cover of some irreducible A -module. In this way one obtains a bijection between the isomorphism classes of the projective indecomposable A -modules and the isomorphism classes of the irreducible A -modules.

Let L be a finite-dimensional restricted Lie algebra over a field \mathbb{F} of prime characteristic, and let $u(L)$ denote the restricted universal enveloping algebra of L (see [19, p. 192] or [24, p. 90]). Then every restricted L -module is an $u(L)$ -module and vice versa, and so there is a bijection between the irreducible restricted L -modules and the irreducible $u(L)$ -modules. In particular, as $u(L)$ is finite-dimensional (see [19, Theorem 12, p. 191] or [24, Theorem 2.5.1(2)]), every irreducible restricted L -module is finite-dimensional. In the sequel we use the notation $P_L(\mathbb{F}) := P_{u(L)}(\mathbb{F})$ for the projective cover of the trivial irreducible L -module. Following Hochschild [15] we define the *restricted cohomology* of L with coefficients in a restricted L -module M by $H_*^n(L, M) := \text{Ext}_{u(L)}^n(\mathbb{F}, M)$ for every non-negative integer n .

Using [3, Proposition 2.4.3] and Theorem 2.1 we obtain a lower bound for the multiplicity of a non-trivial irreducible restricted L -module in the second Loewy layer of $P_L(\mathbb{F})$ (see [25, Theorem 3.7] for the analogue in the modular representation theory of finite groups):

Theorem 5.1. *Let L be a finite-dimensional restricted Lie algebra over a field \mathbb{F} of prime characteristic. Then*

$$[\text{Jac}(u(L))P_L(\mathbb{F}) / \text{Jac}(u(L))^2P_L(\mathbb{F}) : S] \geq [L : S]_{\text{split}}$$

holds for every non-trivial irreducible restricted L -module S .

Proof. Since S is not trivial, it follows from the beginning of Hochschild's six-term exact sequence relating ordinary and restricted cohomology [15, p. 575] (see also [12, Lemma 2(b)]) that $H_*^1(L, S) \cong H^1(L, S)$. Then we obtain from [3, Proposition

2.4.3] and Theorem 2.1 that

$$\begin{aligned} \dim_{\mathbb{F}} \text{End}_L(S) \cdot [\text{Jac}(u(L))P_L(\mathbb{F}) / \text{Jac}(u(L))^2P_L(\mathbb{F}) : S] \\ &= \dim_{\mathbb{F}} \text{Ext}_{u(L)}^1(\mathbb{F}, S) = \dim_{\mathbb{F}} H_*^1(L, S) = \dim_{\mathbb{F}} H^1(L, S) \\ &\geq \dim_{\mathbb{F}} H^1(L, S) - \dim_{\mathbb{F}} H^1(L / \text{Ann}_L(S), S) \\ &= \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}. \end{aligned}$$

Cancelling $\dim_{\mathbb{F}} \text{End}_L(S)$ yields the desired inequality. \square

Remark. If one uses the main result of [23] instead of Theorem 2.1, then the above proof would also work in the case of finite-dimensional modular group algebras. This provides an alternative proof of [25, Theorem 3.7].

Example. Consider the three-dimensional simple Lie algebra $L := \mathfrak{sl}_2(\mathbb{F})$ over an algebraically closed field \mathbb{F} of characteristic $p > 2$. Take for S the $(p-1)$ -dimensional irreducible restricted L -module. Then $[\text{Jac}(u(L))P_L(\mathbb{F}) / \text{Jac}(u(L))^2P_L(\mathbb{F}) : S] = 2$ (see [20, Theorem 1(ii)]), but $[L : S]_{\text{split}} = 0$. This shows that equality does not necessarily hold in Theorem 5.1.

Consider the non-abelian two-dimensional restricted Lie algebra

$$L := \mathbb{F}t \oplus \mathbb{F}e, \quad [t, e] = e, \quad t^{[p]} = t, \quad e^{[p]} = 0.$$

Then it is well known that $P_L(\mathbb{F}) \cong u(L) \otimes_{u(\mathbb{F}t)} \mathbb{F}$ (see [9, Satz II.3.2] and [7, Proposition 2.2]). As the trivial irreducible L -module only occurs in the top of $P_L(\mathbb{F})$, we have that $[\text{Jac}(u(L))P_L(\mathbb{F}) / \text{Jac}(u(L))^2P_L(\mathbb{F}) : \mathbb{F}] = 0$. This in conjunction with Corollary 2.2 shows that Theorem 5.1 is not true for the trivial irreducible L -module.

As an immediate consequence of Theorem 5.1, we obtain the following weak analogue of a well-known result for finite modular group algebras:

Corollary 5.2. *Every non-trivial split chief factor of a finite-dimensional solvable restricted Lie algebra L is a direct summand of the second Loewy layer of the projective cover $P_L(\mathbb{F})$ of the trivial irreducible L -module. In particular, every split chief factor of a finite-dimensional solvable restricted Lie algebra L is a composition factor of $P_L(\mathbb{F})$.*

Recall that the *principal block* of a restricted Lie algebra is the block that contains the trivial irreducible module.

Question. In view of Corollary 5.2, it is natural to ask whether every chief factor of a finite-dimensional solvable restricted Lie algebra L is a composition factor of $P_L(\mathbb{F})$ (for an affirmative answer to the analogous question in the modular representation theory of finite p -solvable groups see [17, Theorem VII.15.8]), or even more generally (see Proposition 5.4 below), whether every irreducible module in the principal block of $u(L)$ is a composition factor of $P_L(\mathbb{F})$.

Let $\langle X \rangle_{\mathbb{F}}$ denote the \mathbb{F} -subspace of L spanned by a subset X of L . Using in addition [1, Theorem 2] and [10, Proposition 2.7] we obtain in the solvable case the following more precise result (for the analogous result of W. Gaschütz in the modular representation theory of finite p -solvable groups see [17, Theorem VII.15.5b]):

Proposition 5.3. *Let L be a finite-dimensional solvable restricted Lie algebra over a field \mathbb{F} of prime characteristic. If S is an irreducible restricted L -module, then*

$$\begin{aligned} & [\text{Jac}(u(L))P_L(\mathbb{F})/\text{Jac}(u(L))^2P_L(\mathbb{F}) : S] \\ &= \begin{cases} [L : S]_{\text{split}} & \text{if } S \not\cong \mathbb{F} \\ [L : S]_{\text{split}} - \dim_{\mathbb{F}}(\langle L^{[p]} \rangle_{\mathbb{F}}/[L, L] \cap \langle L^{[p]} \rangle_{\mathbb{F}}) & \text{if } S \cong \mathbb{F} \end{cases} . \end{aligned}$$

Remark. Note that the second equality in Proposition 5.3 remains true for an arbitrary finite-dimensional restricted Lie algebra. This implies that the reverse inequality of Theorem 5.1 holds for the trivial irreducible module.

For the convenience of the reader we include a proof of the following result which in the case of solvable restricted Lie algebras is already contained in a previous paper of the first author (see [13, Proposition 2]).

Proposition 5.4. *Every abelian chief factor of a finite-dimensional restricted Lie algebra L belongs to the principal block of L .*

Proof. Let $S = I/J$ be an abelian chief factor of L . In particular, S is a trivial I -module. Then the five-term exact sequence for ordinary cohomology (see [16, Theorem 6]) yields

$$0 \rightarrow H^1(L/I, S) \rightarrow H^1(L/J, S) \rightarrow \text{Hom}_L(S, S) \rightarrow H^2(L/I, S) \rightarrow H^2(L/J, S).$$

Since the third term is non-zero, the second or fourth term must also be non-zero. In the first case we obtain from Hochschild's six-term exact sequence relating ordinary and restricted cohomology [15, p. 575] (see also [12, Lemma 2(b)]) and the five-term exact sequence for restricted cohomology that either $S \cong \mathbb{F}$ or $0 \neq H_*^1(L/J, S) \hookrightarrow H_*^1(L, S)$. Therefore S belongs to the principal block of L (see [12, Lemma 1(b)]). In the second case Hochschild's six-term exact sequence relating ordinary and restricted cohomology yields that either $H^1(L, S) \neq 0$ or $H_*^2(L/I, S) \neq 0$. In the former case either $S \cong \mathbb{F}$ or $H_*^1(L, S) \neq 0$, and thus S belongs to the principal block of L . In the latter case S belongs to the principal block of L/I , and by [13, Lemma 4], S also belongs to the principal block of L . \square

Finally, we obtain the following characterization of solvable restricted Lie algebras which was motivated by [25, Theorem 3.9] (see also [11, Remark after Theorem 7] for the equivalence of (i), (vi), and (viii) as well as [12, Theorem 1] for the equivalence of (i), (iv), and (v)).

Theorem 5.5. *Let L be a finite-dimensional restricted Lie algebra over a field \mathbb{F} of prime characteristic. Then the following statements are equivalent:*

- (i) L is solvable.
- (ii) $H^1(L/\text{Ann}_L(S), S) = 0$ for every non-trivial irreducible restricted L -module S .
- (iii) $\dim_{\mathbb{F}} H^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$ holds for every non-trivial irreducible restricted L -module S .
- (iv) $H_*^1(L/\text{Ann}_L(S), S) = 0$ for every non-trivial irreducible restricted L -module S .
- (v) $H_*^1(L/\text{Ann}_L(S), S) = 0$ for every non-trivial irreducible restricted L -module S belonging to the principal block of L .
- (vi) $\dim_{\mathbb{F}} H_*^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$ holds for every non-trivial irreducible restricted L -module S .

- (vii) $\dim_{\mathbb{F}} H_*^1(L, S) = \dim_{\mathbb{F}} \text{End}_L(S) \cdot [L : S]_{\text{split}}$ holds for every non-trivial irreducible restricted L -module S belonging to the principal block of L .
- (viii) $[\text{Jac}(u(L))P_L(\mathbb{F})/\text{Jac}(u(L))^2P_L(\mathbb{F}) : S] = [L : S]_{\text{split}}$ holds for every non-trivial irreducible restricted L -module S .
- (ix) $[\text{Jac}(u(L))P_L(\mathbb{F})/\text{Jac}(u(L))^2P_L(\mathbb{F}) : S] = [L : S]_{\text{split}}$ holds for every non-trivial irreducible restricted L -module S belonging to the principal block of L .

Proof. The equivalence of (i) and (ii) follows from Theorem 4.3 in conjunction with [4, Theorem 2] (see also [14, Corollary 3.2]) and $\text{Ann}_L(\mathbb{F}) = L$. The equivalence of (ii) and (iii) is an immediate consequence of Theorem 2.1. The equivalences of (ii) and (iv) as well as (iii) and (vi) follow from Hochschild's six-term exact sequence relating ordinary and restricted cohomology [15, p. 575] (see also [12, Lemma 2(b)]). Moreover, [13, Lemma 4] and [12, Lemma 1(a)] yield the equivalence of (iv) and (v). This in conjunction with Proposition 5.4 also shows the equivalence of (vi) and (vii). Finally, the equivalences of (vi) and (viii) as well as (vii) and (ix) are both consequences of $\dim_{\mathbb{F}} \text{End}_L(S) \cdot [\text{Jac}(u(L))P_L(\mathbb{F})/\text{Jac}(u(L))^2P_L(\mathbb{F}) : S] = \dim_{\mathbb{F}} H_*^1(L, S)$. \square

It is not surprising that the results in this section do not include the trivial irreducible module. For this it does not suffice to consider abelian chief factors in the category of ordinary Lie algebras, but one has to consider strongly abelian p -chief factors in the category of restricted Lie algebras. We will investigate this in more detail on another occasion.

Acknowledgements. The first and the second author would like to thank the Dipartimento di Matematica e Applicazioni at the Università degli Studi di Milano-Bicocca for the hospitality during their visit in May 2012 when large portions of this paper were written.

REFERENCES

- [1] D. W. Barnes: On the cohomology of soluble Lie algebras, *Math. Z.* **101** (1967), 343–349.
- [2] D. W. Barnes: First cohomology groups of soluble Lie algebras, *J. Algebra* **46** (1977), no. 1, 292–297.
- [3] D. J. Benson: *Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras*, Cambridge Studies in Advanced Mathematics, vol. **30**, Cambridge University Press, Cambridge, 1991.
- [4] A. S. Dzhumadil'daev: On the cohomology of modular Lie algebras, *Math. USSR Sb.* **47** (1984), no. 1, 127–143.
- [5] A. S. Dzhumadil'daev: Cohomology of truncated coinduced representations of Lie algebras of positive characteristic, *Math. USSR Sb.* **66** (1990), no. 2, 461–473.
- [6] A. S. Dzhumadil'daev: Cohomology and nonsplit extensions of modular Lie algebras, in: *Proceedings of the International Conference on Algebra Dedicated to the Memory of A. I. Mal'cev* (eds. L. A. Bokut', Yu. L. Ershov, and A. I. Kostrikin), Contemp. Math., vol. **131**, Part 2, Amer. Math. Soc., Providence, RI, 1992, pp. 31–43.
- [7] R. Farnsteiner: Representations of blocks associated to induced modules of restricted Lie algebras, *Math. Nachr.* **179** (1996), 57–88.
- [8] R. Farnsteiner and H. Strade: Shapiro's lemma and its consequences in the cohomology theory of modular Lie algebras, *Math. Z.* **206** (1991), no. 1, 153–168.
- [9] J. Feldvoss: *Homologische Aspekte der Darstellungstheorie modularer Lie-Algebren*, Doctoral Dissertation, Universität Hamburg, 1989.
- [10] J. Feldvoss: On the cohomology of restricted Lie algebras, *Comm. Algebra* **19** (1991), no. 10, 2865–2906.

- [11] J. Feldvoss: Some conjectures in the block theory of solvable restricted Lie algebras, unpublished manuscript, 1993.
- [12] J. Feldvoss: A cohomological characterization of solvable modular Lie algebras, in: *Non-Associative Algebra and Its Applications* (ed. S. González), Mathematics and Its Applications, vol. **303**, Kluwer Academic Publishers, Dordrecht/Boston/London, 1994, pp. 133–139.
- [13] J. Feldvoss: On the block structure of supersolvable restricted Lie algebras, *J. Algebra* **183** (1996), no. 2, 396–419.
- [14] J. Feldvoss: On the cohomology of modular Lie algebras, in: *Lie Algebras, Vertex Operator Algebras and Their Applications, Raleigh, NC, 2005* (eds. Y.-Z. Huang and K. C. Misra), Contemp. Math., vol. **442**, Amer. Math. Soc., Providence, RI, 2007, pp. 89–113.
- [15] G. Hochschild: Cohomology of restricted Lie algebras, *Amer. J. Math.* **76** (1954), no. 3, 555–580.
- [16] G. Hochschild and J.-P. Serre: Cohomology of Lie algebras, *Ann. of Math. (2)* **57** (1953), no. 3, 591–603.
- [17] B. Huppert and N. Blackburn: *Finite Groups II*, Grundlehren der Mathematischen Wissenschaften, vol. **242**, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
- [18] N. Jacobson: A note on Lie algebras of characteristic p , *Amer. J. Math.* **74**, (1952), no. 2, 357–359.
- [19] N. Jacobson: *Lie Algebras*, Dover Publications, Inc., New York, 1979 (unabridged and corrected republication of the original edition from 1962).
- [20] R. D. Pollack: Restricted Lie algebras of bounded type, *Bull. Amer. Math. Soc.* **74** (1968), no. 2, 326–331.
- [21] G. B. Seligman: *Modular Lie Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **40**, Springer-Verlag, New York, Inc., 1967.
- [22] U. Stammbach: Cohomological characterisations of finite solvable and nilpotent groups, *J. Pure Appl. Algebra* **11** (1977/78), no. 1-3, 293–301.
- [23] U. Stammbach: Split chief factors and cohomology, *J. Pure Appl. Algebra* **44** (1987), no. 1-3, 349–352.
- [24] H. Strade and R. Farnsteiner: *Modular Lie Algebras and Their Representations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. **116**, Marcel Dekker, Inc., New York/Basel, 1988.
- [25] W. Willems: On p -chief factors of finite groups, *Comm. Algebra* **13** (1985), no. 11, 2433–2447.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SOUTH ALABAMA, MOBILE, AL 36688-0002, USA

E-mail address: jfeldvoss@southalabama.edu

DIPARTIMENTO DI MATEMATICA E FISICA “ENNIO DE GIORGI”, UNIVERSITÀ DEL SALENTO, VIA PROVINCIALE LECCE-ARNESANO, I-73100 LECCE, ITALY

E-mail address: salvatore.siciliano@unisalento.it

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA ROBERTO COZZI, NO. 53, I-20125 MILANO, ITALY

E-mail address: thomas.weigel@unimib.it