OUTER RESTRICTED DERIVATIONS OF NILPOTENT RESTRICTED LIE ALGEBRAS

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Abstract. In this paper we prove that every finite-dimensional nilpotent restricted Lie algebra over a field of prime characteristic has an outer restricted derivation whose square is zero unless the restricted Lie algebra is a torus or it is one-dimensional or it is isomorphic to the three-dimensional Heisenberg algebra in characteristic two as an ordinary Lie algebra. This result is the restricted analogue of a result of Tôgô on the existence of nilpotent outer derivations of ordinary nilpotent Lie algebras in arbitrary characteristic and the Lie-theoretic analogue of a classical group-theoretic result of Gaschütz on the existence of p-power automorphisms of p-groups. As a consequence we obtain that every finite-dimensional non-toral nilpotent restricted Lie algebra has an outer restricted derivation.

1. Introduction

In 1966 W. Gaschütz proved the following celebrated result:

Theorem. (W. Gaschütz [3]) *Every finite p-group of order $> p$ has an outer automorphism of p-power order.*

Since every finite nilpotent group is a direct product of its Sylow p-subgroups, the outer automorphism group of a finite nilpotent group is a direct product of the outer automorphism groups of its Sylow p-subgroups. Therefore it is a direct consequence of Gaschütz' theorem that every finite nilpotent group has an outer automorphism. This answers a question raised by E. Schenkman and F. Haimo in the affirmative (see [9]).

Since groups and Lie algebras often have structural properties in common, it seems rather natural to ask whether an analogue of Gaschütz' theorem holds in the setting of ordinary or restricted Lie algebras. In the case of ordinary Lie algebras a stronger version of such an analogue is already known and was established by S. Tôgô around the same time as Gaschütz proved his result.

Theorem. (S. Tôgô [11, Corollary 1]) *Every nilpotent Lie algebra of finite dimension > 1 over an arbitrary field has an outer derivation whose square is zero.*

In fact, Tôgô's result is more general (see [11, Theorem 1]) and is a refinement of a theorem of E. Schenkman that establishes the existence of outer derivations for non-zero finite-dimensional nilpotent Lie algebras (see [7, Theorem 4]). Much later the first author proved a restricted analogue of Schenkman's result for p-unipotent

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restricted Lie algebras (see [2, Corollary 5.2]). (Here we follow [1, Section I.4, Exercise 23, p. 97] by calling a restricted Lie algebra $(L,[p])$ *p-unipotent* if for every $x \in L$ there exists some positive integer n such that $x^{[p]^n} = 0$.) In this paper we prove that every finite-dimensional nilpotent restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$ has an outer restricted derivation whose square is zero unless L is a torus or $\dim_{\mathbb{F}} L = 1$ or $p = 2$ and L is isomorphic to the three-dimensional Heisenberg algebra $\mathfrak{h}_1(\mathbb{F})$ over $\mathbb{F}$ as an ordinary Lie algebra (see Theorem 1). Indeed, in the later three cases every nilpotent restricted derivation is inner. As a consequence we also obtain a generalization of [2, Corollary 5.2] to non-toral nilpotent restricted Lie algebras (see Theorem 2) which is the full analogue of Schenkman's result for nilpotent restricted Lie algebras.

In the following we briefly recall some of the notation that will be used in this paper. A derivation D of a restricted Lie algebra L is called *restricted* if $D(x^{[p]}) = (\operatorname{ad}_L x)^{p-1}(D(x))$ for every $x \in L$ (see [6, Section 4, (15), p. 21]) and the set of all restricted derivations of L is denoted by $\operatorname{Der}_p(L)$. Observe that $\operatorname{Der}_p(L)$ is a restricted Lie algebra (see [6, Theorem 4]) and that every inner derivation of L is restricted. In this paper we will use frequently without further explanation that the vector space of outer restricted derivations $\operatorname{Der}_p(L)/\operatorname{ad}(L)$ of L is isomorphic to the *first adjoint restricted cohomology space* $H^1_*(L,L)$ of L in the sense of Hochschild (see [4, Theorem 2.1]). More generally, we will need for any restricted L-module M the vector space

$$Z^1_*(L,M) := \{\, D \in \operatorname{Der}(L,M) \mid \forall\, x \in L : D(x^{[p]}) = (x)_M^{p-1}(D(x)) \,\}$$

of *restricted* 1-*cocycles* of L with values in M (see again [4, Theorem 2.1]) and the vector space

$$B^1_*(L,M) := \{\, D \in \operatorname{Hom}_{\mathbb{F}}(L,M) \mid \exists\, m \in M\, \forall\, x \in L : D(x) = x \cdot m \,\}$$

of *restricted* 1-*coboundaries* of L with values in M. Then

$$H^1_*(L,M) := Z^1_*(L,M)/B^1_*(L,M)$$

denotes the *first restricted cohomology space* of L with values in M. For any L-module M we denote by $M^L := \{\, m \in M \mid \forall\, x \in L : x \cdot m = 0 \,\}$ the *space of L-invariants* of M and by $\operatorname{Soc}_L(M)$ the largest semisimple L-submodule of M. Moreover, $u(L)$ denotes the *restricted universal enveloping algebra* of L (see [6, Section 2], [8, Section V.7, Theorem 12], or [10, Section 2.5]). For a subset S of L we denote by $\langle S \rangle_p$ the *p-subalgebra* of L *generated by* S and by $\operatorname{Cent}_L(S)$ the *centralizer* of S in L. Finally, $\mathrm{Z}(L) := \operatorname{Cent}_L(L)$ is the *center* of L and L' is the *derived subalgebra* of L. For more notation and well-known results from the structure theory of ordinary and restricted Lie algebras we refer the reader to the first two chapters in [10].

2. Preliminaries

2.1. **Restricted derivations and cohomology.** Let L be a restricted Lie algebra over a field of characteristic $p > 0$ and let I be a p-ideal of L. Then the centralizer $\operatorname{Cent}_L(I) = L^I$ of I in L is a p-ideal of L. By virtue of the five-term exact sequence associated to the Hochschild-Serre spectral sequence the canonical mapping

$$\iota : H^1_*(L/I, \operatorname{Cent}_L(I)) \longrightarrow H^1_*(L,L)$$

is injective. Hence one has the following sufficient condition for the existence of outer restricted derivations:

Lemma 1. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$, let I be a p-ideal of L, and let D be a restricted derivation of L satisfying $I \subseteq \mathrm{Ker}(D)$ as well as $\mathrm{Im}(D) \subseteq \mathrm{Cent}_L(I)$, but $\mathrm{Im}(D) \not\subseteq [L, \mathrm{Cent}_L(I)]$. Then D is not inner.*

Proof. Let $\tilde{D} \in Z^1_*(L/I, \mathrm{Cent}_L(I))$ be the linear transformation induced by D. By hypothesis, $\mathrm{Im}(\tilde{D}) \not\subseteq [L, \mathrm{Cent}_L(I)]$ and thus $\tilde{D} \notin B^1_* := B^1_*(L/I, \mathrm{Cent}_L(I))$. Hence $\tilde{D} + B^1_* \neq 0$, and therefore $D + \mathrm{ad}(L) = \iota(\tilde{D} + B^1_*) \neq 0$, i.e., D is not inner. □

By using again the injectivity of ι one deduces the following criterion for the existence of outer restricted derivations whose square is zero.

Lemma 2. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$ that contains two p-ideals I and J with the following properties:*

(i) *$J \subseteq I \cap \mathrm{Cent}_L(I) = \mathrm{Z}(I)$ and*
(ii) *the canonical mapping $\gamma : H^1_*(L/I, J) \to H^1_*(L/I, \mathrm{Cent}_L(I))$ is non-trivial.*

Then there exists an outer restricted derivation of L whose square is zero.

Proof. Let $\tilde{D} \in Z^1_*(L/I, J)$ be such that $\gamma(\tilde{D} + B^1_*(L/I, J)) \neq 0$ and let $D \in Z^1_*(L, L)$ be the linear transformation induced by $\tilde{D}$. Then it follows from

$$\mathrm{Im}(D) \subseteq J \subseteq I \subseteq \mathrm{Ker}(D)$$

that $D^2 = 0$. Moreover,

$$D + \mathrm{ad}(L) = \iota(\gamma(\tilde{D} + B^1_*(L/I, J))) \neq 0\,,$$

i.e., D is an outer restricted derivation of L with $D^2 = 0$. □

2.2. **Abelian p-ideals.** In the proof of our main result we will need the following criterion for the existence of certain p-supplements of abelian p-ideals (for the first part of the proof see also [4, Lemma 3.1]).

Proposition 1. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$ and let A be an abelian p-ideal of L containing $\mathrm{Z}(L)$. If $H^2_*(L/A, A) = 0$, then there exists a p-subalgebra H of L such that $L = A + H$ and $A \cap H = \mathrm{Z}(L)$.*

Proof. Let $\mathcal{B}_A$ be a vector space basis of A being contained in a vector space basis $\mathcal{B}$ of L. According to [8, Section V.7, Theorem 11] or [10, Theorem 2.2.3], the function $\psi : \mathcal{B} \to L$ defined by

$$\psi(x) := \begin{cases} 0 & \text{if } x \in \mathcal{B}_A \\ x^{[p]} & \text{if } x \in \mathcal{B} \setminus \mathcal{B}_A \end{cases}$$

can be extended uniquely to a p-mapping $\_^{[p]'} : L \to L$ making $(L, [p]')$ a restricted Lie algebra. By construction,

$$x^{[p]} - x^{[p]'} \in \mathrm{Z}(L) \quad \text{for every } x \in L\,. \tag{1}$$

The ideal A – with trivial p-mapping – is also a p-ideal of $(L, [p]')$, and the identity mapping induces an isomorphism between $(L/A, [p])$ and $(L/A, [p]')$. It follows from $H^2_*(L/A, A) = 0$ that there exists a p-subalgebra C of $(L, [p]')$ such that $L = A \oplus C$ (see [4, Theorem 3.3]). Then by (1), $H := \mathrm{Z}(L) \oplus C$ is a p-subalgebra of $(L, [p])$ with the desired properties. □

Remark. A similar proof shows that Proposition 1 holds more generally for abelian p-ideals A of L that not necessarily contain the center of L if one replaces everywhere $\mathrm{Z}(L)$ by the image of A under the p-mapping of $(L,[p])$.

The next result shows that maximal abelian p-ideals of finite-dimensional non-abelian nilpotent restricted Lie algebras are self-centralizing and contain the center properly.

Proposition 2. *Let L be a finite-dimensional non-abelian nilpotent restricted Lie algebra over a field of characteristic $p>0$ and let A be a maximal abelian p-ideal of L. Then $\mathrm{Z}(L)\subsetneq A=\mathrm{Cent}_L(A)$.*

Proof. Suppose by contradiction that $C:=\mathrm{Cent}_L(A)\supsetneq A$. As C/A is a non-zero ideal of the nilpotent Lie algebra L/A, there is an element x in L that does not belong to A such that $x+A\in\mathrm{Z}(L/A)\cap C/A$. Put $X:=A+\langle x\rangle_p$. Then X is an abelian p-ideal of L properly containing A which by the maximality of A implies that $L=X$ is abelian contradicting the hypothesis that L is non-abelian.

It follows from $A=\mathrm{Cent}_L(A)$ that A is a faithful L/A-module under the induced adjoint action. Suppose now that $\mathrm{Z}(L)=A$. Then $\mathrm{Z}(L)$ is a faithful and trivial $L/\mathrm{Z}(L)$-module under the induced adjoint action which implies that $L=\mathrm{Z}(L)$ is abelian, which again is a contradiction. □

Let $\mathrm{T}_p(L)$ denote the set of all semisimple elements of a finite-dimensional nilpotent restricted Lie algebra L. Since L is nilpotent, we have that $\mathrm{T}_p(L)\subseteq\mathrm{Z}(L)$ and thus $\mathrm{T}_p(L)$ is the unique maximal torus of L. Furthermore, $\mathrm{T}_p(L)$ is a p-ideal of L and it follows from [10, Theorem 2.3.4] that L/I is p-unipotent for every p-ideal I of L that contains $\mathrm{T}_p(L)$. These well-known results will be used in the following proofs without further explanation.

Proposition 3. *Let L be a finite-dimensional non-abelian nilpotent restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$ and let A be a maximal abelian p-ideal of L. If furthermore every restricted derivation D of L with $D^2=0$ is inner, then A is a free $u(L/A)$-module of rank $r:=\dim_{\mathbb{F}}\mathrm{Z}(L)$. In particular, $\dim_{\mathbb{F}}L=d+r\cdot p^d$, where $d:=\dim_{\mathbb{F}}L/A$.*

Proof. Suppose that $H^1_*(L/A,A)\neq 0$. Then by Lemma 2 (applied for $I:=J:=A$) in conjunction with Proposition 2, there exists an outer restricted derivation of L whose square is zero, a contradiction. Hence $H^1_*(L/A,A)=0$.

Another application of Proposition 2 yields $\mathrm{T}_p(L)\subseteq A$. Consequently, L/A is p-unipotent and it follows from [2, Proposition 5.1] that A is a free $u(L/A)$-module. Moreover, by virtue of the Engel-Jacobson theorem (see [10, Corollary 1.3.2(1)]), the trivial L/A-module $\mathbb{F}$ is the only irreducible restricted $u(L/A)$-module up to isomorphism. Thus

$$\mathrm{Soc}_{L/A}(A)=A^{L/A}=\mathrm{Z}(L)\,,$$

and A is a free $u(L/A)$-module of rank $r:=\dim_{\mathbb{F}}\mathrm{Z}(L)$. In particular, $\dim_{\mathbb{F}}A=r\cdot p^d$ and the dimension formula for L follows. □

2.3. **Maximal p-ideals of nilpotent restricted Lie algebras.** For the convenience of the reader we include a proof of the following result.

Lemma 3. *Let L be a finite-dimensional non-toral nilpotent restricted Lie algebra over a field of characteristic $p>0$. Then every maximal p-ideal of L containing the unique maximal torus of L has codimension one in L.*

Proof. Let I be a maximal p-ideal of L that contains the unique maximal torus $\mathrm{T}_p(L)$ of L. Then $\mathrm{Z}(L/I) \neq 0$ and because L/I is p-unipotent, there exists a non-zero central element z of L/I such that $z^{[p]} = 0$. Then $Z := \mathbb{F}z$ is a one-dimensional p-ideal of L/I and the inclusion-preserving bijection between p-ideals of L/I and p-ideals of L that contain I in conjunction with the maximality of I yields that $L/I = Z$ is one-dimensional. □

3. Main results

The main goal of this paper is to establish the existence of nilpotent outer restricted derivations of finite-dimensional nilpotent restricted Lie algebras. It is well-known that the restricted cohomology of tori vanishes (see [4] and the main result of [5] or [2, Corollary 3.6]). Hence tori have no outer restricted derivations. For the convenience of the reader we include the following straightforward proof of the later statement.

Proposition 4. *Every restricted derivation of a finite-dimensional torus over any field of characteristic $p > 0$ vanishes.*

Proof. Suppose that L is a torus and let D be any restricted derivation of L. Then we have $D(x^{[p]}) = (\mathrm{ad}_L\, x)^{p-1}(D(x)) = 0$ for every $x \in L$, and inductively, $D(x^{[p]^n}) = 0$ for every $x \in L$ and every positive integer n. Since $x \in \langle x^{[p]} \rangle_p$, we conclude that $D(x) = 0$ for every $x \in L$. □

If L is one-dimensional, then either L is a torus or $L = \mathbb{F}e$ with $e^{[p]} = 0$. In the second case the (outer) restricted derivations of L coincide with the vector space endomorphisms $\mathrm{End}_{\mathbb{F}}(L) = \mathbb{F} \cdot \mathrm{id}_L$ of L and therefore no non-zero (outer) restricted derivation is nilpotent.

Indeed there is only one more finite-dimensional nilpotent restricted Lie algebra (up to isomorphism of ordinary Lie algebras) that has no nilpotent outer restricted derivations, namely the three-dimensional restricted Heisenberg algebra over a field of characteristic two. Since the derived subalgebra of any Heisenberg algebra is central, it follows from [10, Example 2, p. 72] that Heisenberg algebras are restrictable. For the proof of the next result one only needs that the image of every p-mapping is central so that the result does not depend on the particular choice of the p-mapping.

Proposition 5. *Every nilpotent restricted derivation of a three-dimensional restricted Heisenberg algebra over a field of characteristic two is inner.*

Proof. Let $\mathbb{F}$ be any field of characteristic 2 and let L be the three-dimensional Heisenberg algebra $\mathfrak{h}_1(\mathbb{F}) = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ defined by $[x, y] = z \in \mathrm{Z}(L)$. Since L' is central, we have that

$$L^{[2]} \subseteq \mathrm{Z}(L) = \mathbb{F}z\,. \tag{2}$$

Suppose that D is any nilpotent outer restricted derivation of L. It follows from

$$[x, D(z)] = [y, D(z)] = 0$$

that z is an eigenvector of D. This forces $D(z) = 0$, as D is nilpotent. Moreover, by (2) we obtain that

$$(\mathrm{ad}_L\, x)(D(x)) = D(x^{[2]}) = 0\,,$$
$$(\mathrm{ad}_L\, y)(D(y)) = D(y^{[2]}) = 0\,,$$

and then

$$D(x) = \alpha x + \lambda z\,,$$
$$D(y) = \beta y + \mu z$$

for suitable $\alpha, \beta, \lambda, \mu \in \mathbb{F}$. Furthermore, $D([x, y]) = 0$ yields $\alpha = \beta$.

Now, observe that $D^2 = \alpha D$ and therefore $D^n = \alpha^{n-1} D$ for every positive integer n. As D is nilpotent, we conclude that $\alpha = 0$. Thus one has

$$D = \mathrm{ad}_L(\lambda y + \mu x)\,,$$

so that D is inner, as claimed. □

Now we are ready to prove our first main result:

Theorem 1. *Let $(L, [p])$ be a nilpotent restricted Lie algebra of finite dimension > 1 over a field $\mathbb{F}$ of characteristic $p > 0$. Then L has an outer restricted derivation D with $D^2 = 0$ unless L is a torus or $p = 2$ and $L \cong \mathfrak{h}_1(\mathbb{F})$ as ordinary Lie algebras.*

Proof. Assume that L is neither a torus nor one-dimensional. In the following T will denote the unique maximal torus $\mathrm{T}_p(L)$ of L and by assumption $L/T \neq 0$. We proceed by a case-by-case analysis.

Case 1: *There exists a maximal p-ideal I of L containing T such that* $\mathrm{Z}(L) \not\subseteq I$.
According to Lemma 3, I has codimension 1 in L and therefore $L = \mathbb{F}x \oplus I$ for any $x \in \mathrm{Z}(L) \backslash I$. Let $0 \neq z \in \mathrm{Z}(I)$ and consider the linear transformation D of L defined by setting $D(x) := z$ and $D(y) := 0$ for every $y \in I$. Then D is a derivation of L (see [1, Section I.4, Exercise 8(a), p. 92]) with $D^2 = 0$. Moreover, the inner derivation $\mathrm{ad}_L\, a$ vanishes on x for every $a \in L$ and thus D cannot be inner. Finally, as L/I is p-unipotent and I has codimension 1 in L, it follows that $L^{[p]} \subseteq I$. This implies that $D(l^{[p]}) = 0 = (\mathrm{ad}_L\, l)^{p-1}(D(l))$ for every $l \in L$, and so D is restricted.

Note that Case 1 covers already the abelian case. So for the rest of the proof we may assume that L is non-abelian.

Case 2: $\mathrm{Z}(L) \subseteq I$ *for every maximal p-ideal I of L containing T.*
Suppose that $\mathrm{Cent}_L(I) \not\subseteq I$ for some maximal p-ideal I of L containing T. Then there exists $x \in \mathrm{Cent}_L(I) \backslash I$ such that $L = \mathbb{F}x \oplus I$; in particular, $x \in \mathrm{Z}(L) \subseteq I$, a contradiction. Hence we may assume from now on that $\mathrm{Cent}_L(I) = \mathrm{Z}(I)$ for every maximal p-ideal I of L containing T.

Case 2.1: *There exists a maximal p-ideal I of L containing T such that* $\mathrm{Cent}_L(I)$ $= \mathrm{Z}(L)$.
Let $x \in L \backslash I$ be such that $L = \mathbb{F}x \oplus I$. Let $0 \neq z \in \mathrm{Z}(L)$ and consider the vector space endomorphism D of L given by $D(x) := z$ and $D(y) := 0$ for every $y \in I$. Then as above, D is a restricted derivation of L with $D^2 = 0$. By construction, $\mathrm{Ker}(D) = I$ and $\mathrm{Im}(D) = \mathbb{F}z$. Since by hypothesis $[L, \mathrm{Cent}_L(I)] = 0$, it follows from Lemma 1 that D is not inner.

Case 2.2: $\mathrm{Z}(L) \subsetneq \mathrm{Cent}_L(I) = \mathrm{Z}(I)$ *for every maximal p-ideal I of L containing T.*
Suppose that every restricted derivation D of L satisfying $D^2 = 0$ is inner. Let A be a maximal abelian p-ideal of L. In particular, since L is not abelian, $L/A \neq 0$, and in view of Proposition 2, A contains $\mathrm{Z}(L)$ properly. According to Proposition 3, A

is a free $u(L/A)$-module of rank $r := \dim_{\mathbb{F}} \mathrm{Z}(L)$. In particular, $\dim_{\mathbb{F}} L = d + r \cdot p^d$, where $d := \dim_{\mathbb{F}} L/A$.

Since A is a free $u(L/A)$-module, [2, Proposition 5.1] implies that $H^2_*(L/A, A) = 0$. Hence by Proposition 1, there exists a p-subalgebra H of L such that $L = A + H$ and $A \cap H = \mathrm{Z}(L)$. In particular, H contains T. As A contains $Z(L)$ properly, one has $H \neq L$. Choose now a proper p-subalgebra J of L with $H \subseteq J$ of maximal dimension. Then J is a maximal p-subalgebra of L. Since J is properly contained in L, the nilpotency of L implies that J is properly contained in the p-subalgebra $N_L(J) := \{ x \in L \mid [x, J] \subseteq J \}$. Hence the maximality of J yields $N_L(J) = L$, i.e., J is a maximal p-ideal of L containing H and in particular T. According to Lemma 3, J is of codimension 1 in L. Since $L = A + J$, one has

$$\dim_{\mathbb{F}} A/A \cap J = \dim_{\mathbb{F}} A + J/J = \dim_{\mathbb{F}} L/J = 1\,,$$

i.e., $A \cap J$ has codimension 1 in A.

Let $B := \mathrm{Cent}_L(J) = \mathrm{Z}(J)$. By hypothesis, there exists an element $e \in B \setminus \mathrm{Z}(L)$. We obtain from $B \subseteq \mathrm{Cent}_L(H)$ and $L = A + H$ that $B \cap A = \mathrm{Z}(L)$ which implies that $e \notin A$. Since B is an abelian p-ideal of L, $e^{[p]} \in \mathrm{Z}(L) \subseteq A$. Hence it follows from [10, Lemma 2.1.2] that $\mathbb{F}e \oplus A$ is a p-subalgebra of L.

Let $E := \mathbb{F}e \oplus A/A$ be the one-dimensional p-unipotent p-subalgebra generated by the image of e in L/A. Note that $A \cap J \subseteq A^E$ and $\dim_{\mathbb{F}} A \cap J = r \cdot p^d - 1$. By construction, A is a free $u(E)$-module of rank $r \cdot p^{d-1}$. Consequently,

$$r \cdot p^d - 1 = \dim_{\mathbb{F}} A \cap J \leq \dim_{\mathbb{F}} A^E = \dim_{\mathbb{F}} \mathrm{Soc}_E(A) = r \cdot p^{d-1}\,.$$

But this implies $p^d = 2$ and $r = 1$. Hence $\mathbb{F}$ has characteristic 2 and $\dim_{\mathbb{F}} L = 3$. Since the Heisenberg algebra is the only non-abelian nilpotent three-dimensional Lie algebra up to isomorphism (see [8, Section I.4, pp. 11–13], [10, Section 1.6, p. 34], or [1, Section I.4., Exercise 9(c), p. 92]), this finishes the proof. □

Theorem 1 in conjunction with the introductory remarks of this section yields a characterization of finite-dimensional nilpotent restricted Lie algebras in terms of the existence of nilpotent outer restricted derivations.

Corollary 1. *For a finite-dimensional nilpotent restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$ the following statements are equivalent:*

- (i) *L is a torus or $\dim_{\mathbb{F}} L = 1$ and $L^{[p]} = 0$ or $p = 2$ and $L \cong \mathfrak{h}_1(\mathbb{F})$ as ordinary Lie algebras.*
- (ii) *L has no nilpotent outer restricted derivation.*
- (iii) *L has no outer restricted derivation D with $D^2 = 0$.*

Proof. Clearly, (ii) implies (iii) and it follows from Theorem 1 that (iii) implies (i). It remains to prove the implication (i)$\Longrightarrow$(ii) which is an immediate consequence of Proposition 4, the paragraph thereafter, and Proposition 5. □

If we exclude the one-dimensional case and assume that the characteristic of the ground field is greater than two, then Theorem 1 can be used to characterize the tori among nilpotent restricted Lie algebras in terms of the non-existence of restricted derivations with various nilpotency properties.

Corollary 2. *For a nilpotent restricted Lie algebra L of finite dimension > 1 over a field of characteristic $p > 2$ the following statements are equivalent:*

- (i) *L is a torus.*

(ii) *L has no non-zero restricted derivation.*
(iii) *L has no nilpotent outer restricted derivation.*
(iv) *L has no outer restricted derivation D with $D^2 = 0$.*

Proof. Obviously, the implications (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv) hold. The implications (iv)$\Longrightarrow$(i) and (i)$\Longrightarrow$(ii) are immediate consequences of Theorem 1 and Proposition 4, respectively. □

Finally, we obtain from Theorem 1 the following generalization of [2, Corollary 5.2]:

Theorem 2. *Every finite-dimensional non-toral nilpotent restricted Lie algebra over a field of characteristic $p > 0$ has an outer restricted derivation.*

Proof. According to Theorem 1 and the first paragraph after Proposition 4, it is enough to show the assertion for the three-dimensional restricted Heisenberg algebra $L := \mathfrak{h}_1(\mathbb{F})$ over a field $\mathbb{F}$ of characteristic 2. Denote the one-dimensional center of L by Z. If L is 2-unipotent, then the claim follows from [2, Corollary 5.2]. Suppose now that L is not 2-unipotent. Then the center Z of L is a one-dimensional torus. According to [2, Proposition 3.9] in conjunction with [2, Corollary 3.6], we obtain that $H^1_*(L, L) \cong H^1_*(L/Z, L)$. Since L' is central, we have $L^{[2]} \subseteq Z$, and consequently, L/Z is 2-unipotent. Suppose now that L has no outer restricted derivation and therefore $H^1_*(L/Z, L) \cong H^1_*(L, L) = 0$. Then an application of [2, Proposition 5.1] yields that L is a free $u(L/Z)$-module and it follows that

$$3 = \dim_{\mathbb{F}} L \geq \dim_{\mathbb{F}} u(L/Z) = 4\,,$$

a contradiction. □

Neither Theorem 1 nor Theorem 2 does generalize to non-toral *solvable* restricted Lie algebras with *non-zero center* as the direct product of a two-dimensional non-abelian restricted Lie algebra and a one-dimensional torus over any field of characteristic $p > 0$ shows. In this case it is not difficult to see that there are no outer restricted derivations. Consequently, in general there is no restricted analogue of Tôgô's result [11, Corollary 1] for solvable Lie algebras and also [11, Theorem 1] has no analogue for restricted Lie algebras.

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