On the Number of Simple Modules of a Supersolvable Restricted Lie Algebra

Jörg Feldvoss*
Mathematisches Seminar der Universität
Bundesstrasse 55, D-20146 Hamburg
Germany

§0. Around 1939 Hans Zassenhaus started to study the representation theory of finite-dimensional Lie algebras over a field of prime characteristic [18, 19]. Since then a lot of progress has been made but nevertheless a classification of the simple modules (up to isomorphism) is not known for many classes of Lie algebras. In the papers cited above H. Zassenhaus gave such a classification for nilpotent Lie algebras. About 30 years later parts of this were extended to supersolvable Lie algebras by B. Yu. Veisfeiler and V. G. Kac (see [17, §2]). More generally, they introduced a partition of the (infinite) set of isomorphism classes of simple modules into (infinitely many) *finite* sets, namely, the sets of isomorphism classes of simple modules with a fixed p-character (see [17, §1]).

In particular, for a nilpotent restricted Lie algebra the number of isomorphism classes of simple modules with a fixed p-character is known (see [15, Satz 6]). A fundamental question that still remains open is to determine this number for more general classes of Lie algebras.

The aim of this paper is to develop an approach for attacking this problem in the case of supersolvable (restricted) Lie algebras. In the following we will describe the contents of the paper in more detail.

The first section provides the framework for the paper by considering a decomposition of a reduced universal enveloping algebra into a direct sum of certain two-sided ideals. We show that all these ideals are Frobenius algebras of the same dimension (see Theorem 2). Their simple modules are simple modules with a fixed eigenvalue function (of the largest toral ideal) and the number of their isomorphism classes is a p-power. Moreover, this induces a partition of the set of isomorphism classes of simple modules with a fixed p-character (see Proposition 1). All this was motivated in order to overcome the problems which occurred in [7] where instead the decomposition into block ideals was used to get insight into the simple modules with a fixed p-character. There the group action did *not* leave the equivalence classes of simple modules invariant (see [9, Example 1]); here this follows by definition and makes everything work well.

In the second section we decompose certain induced projective modules into indecomposables, and thereby we obtain a generalization of [8, Theorem 1(b)] from the nilpotent to the supersolvable case (see Theorem 3). This will be further generalized in [11, Theorem 1] by investigating induced modules of projective covers for arbitrary polarizations.

The last section is devoted to show how Theorem 3 can be applied to derive from the dimension formula for projective indecomposable modules (see [4, Corollary 2.10(b)] or [11, Corollary 1]) a formula for the number of the isomorphism classes of simple modules with a fixed p-character *and* a fixed eigenvalue function (see Proposition 2). In particular, this enables us to obtain an upper bound

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for the number of isomorphism classes of all simple modules with a fixed p-character, and a lower bound is an immediate consequence of the partition into simple modules with a fixed eigenvalue function (see Theorem 4). Finally, we illustrate our methods by considering the special case of a *strongly solvable* restricted Lie algebra (see Theorem 5) and an example showing that the number of isomorphism classes of simple modules with a fixed p-character is *not* necessarily a p-power. In particular, this implies that the formula in [2, Corollary 1 of Theorem 4, p. 215] is not valid and that the bounds in Theorem 4 are not the best possible.

§1. In this paper let $\mathbb{F}$ always be a field of prime characteristic p. If L is a finite-dimensional restricted Lie algebra over $\mathbb{F}$, we are interested in the category of finite-dimensional (unitary, left) $u(L,\chi)$-modules for an arbitrary p-character $\chi \in L^* := \mathrm{Hom}_{\mathbb{F}}(L,\mathbb{F})$ (cf. [17, §1] or [16, Chapter 5, Sections 2 and 3]).

A Lie algebra L is called *supersolvable* if there is a (descending) chain

$$L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$$

of ideals L_j in L such that the factor algebras L_j/L_{j+1} are one-dimensional for every $0 \leq j \leq n-1$. It is well-known that subalgebras and factor algebras of supersolvable Lie algebras are again supersolvable.

Consider the (commutative p-)subgroup

$$G^L := \{\gamma \in L^* \mid \gamma([L,L]) = 0, \gamma(x^{[p]}) = \gamma(x)^p \ \ \forall\, x \in L\}$$

of the (additive) group L^* (cf. [16, p. 242]). As a consequence of the Jordan-Chevalley-Seligman-Schue decomposition, G^L is finite (see [16, Proposition 5.8.8(1)]). For every $\gamma \in G^L$ the one-dimensional vector space $F_\gamma := \mathbb{F} \cdot 1_\gamma$ is a restricted L-module via $x \cdot 1_\gamma := \gamma(x) \cdot 1_\gamma$, and conversely, every one-dimensional restricted L-module occurs in this way.

Let $\mathrm{Irr}(L,\chi)$ denote the set of isomorphism classes of simple $u(L,\chi)$-modules. Then for every $\chi \in L^*$ the group G^L acts on $\mathrm{Irr}(L,\chi)$ via

$$\gamma \cdot [S] := [F_\gamma \otimes_{\mathbb{F}} S]$$

(see [16, Proposition 5.8.8(2)]).

If $C(L)$ denotes the *center* of L, then

$$T_p(L) := \{x \in C(L) \mid x \text{ is semisimple }\}$$

is the largest toral ideal of L, and it is obvious that

$$G_0^L := \{\gamma \in G^L \mid \gamma_{|T_p(L)} = 0\}$$

is a subgroup of G^L. Note that by virtue of [9, Theorem 5] for *supersolvable* restricted Lie algebras this definition of G_0^L coincides with the definition in [9, p. 411]. Finally, set

$$\Pi(L,\chi) := \{\tau \in T_p(L)^* \mid \tau(t)^p - \tau(t^{[p]}) = \chi(t)^p \ \ \forall\, t \in T_p(L)\}.$$

Let S be a simple L-module and assume that the ground field $\mathbb{F}$ is algebraically closed. Since $T_p(L) \subseteq C(L)$ and S is simple, Schur's lemma implies that for every $t \in T_p(L)$ there exists an element $\sigma(t) \in \mathbb{F}$ with $(t)_S = \sigma(t) \cdot \mathrm{id}_S$. If S is a $u(L,\chi)$-module, then $\sigma \in \Pi(L,\chi)$ (and conversely), i.e., F_σ is a $u(T_p(L), \chi_{|T_p(L)})$-module and

$$S_{|T_p(L)} \cong F_\sigma^{\oplus \dim_{\mathbb{F}} S}.$$

Since σ is uniquely determined by S, we call σ the *eigenvalue function* of S.

The following version of the equivalence (a)$\Longleftrightarrow$(c) in [9, Theorem 3] will be crucial for this paper. Instead of deriving this from [9, Theorem 3] by applying [9, Theorem 5 and Theorem 1], we will choose here a direct approach which does not use any results from block theory or changing the p-map as in the proof of [16, Theorem 5.8.7(1)].

Theorem 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$, let $\chi \in L^*$, and let M, N be simple $u(L,\chi)$-modules. Then the following statements are equivalent:*

(a) *M and N have the same eigenvalue function.*

(b) $M_{|T_p(L)} \cong N_{|T_p(L)}$.

(c) *There exists an element $\gamma \in G_0^L$ such that $N \cong F_\gamma \otimes_{\mathbb{F}} M$.*

Proof. The implication (c)$\Longrightarrow$(b) is an immediate consequence of the definition of G_0^L and the implication (b)$\Longrightarrow$(a) is trivial. Hence it remains to show the implication (a)$\Longrightarrow$(c).

According to [16, Theorem 5.2.7(1)], $X := \mathrm{Hom}_{\mathbb{F}}(M,N)$ is a finite-dimensional restricted L-module. Since by hypothesis M and N have the same eigenvalue function, $Y := \mathrm{Hom}_{T_p(L)}(M,N) = X^{T_p(L)}$ is a non-zero L-submodule of X. Hence Y has a non-zero socle, i.e., there exists a simple L-submodule S of Y. In particular, S is a simple restricted $L/T_p(L)$-module. By the argument in the proof of [9, Theorem 6], $L/T_p(L)$ is a semidirect product of a torus and a p-nilpotent restricted Lie algebra, and thus $\dim_{\mathbb{F}} S = 1$. As a result, there exists an element $\gamma \in G_0^L$ such that $S \cong F_\gamma$ (as L-modules). Finally, we obtain from the adjointness of Hom and $\otimes$ that

$$\mathrm{Hom}_L(F_\gamma \otimes_{\mathbb{F}} M, N) \cong \mathrm{Hom}_L(F_\gamma, X) \neq 0.$$

Since M and N are simple and F_γ is one-dimensional, Schur's lemma yields $N \cong F_\gamma \otimes_{\mathbb{F}} M$. □

Remark. If in Theorem 1 L is assumed to be *nilpotent*, then $L/T_p(L)$ is *p-nilpotent* and the simple restricted $L/T_p(L)$-module S in the above proof is *trivial*. Hence by the same argument one can conclude that $M \cong N$. This result goes back to H. Zassenhaus [18, p. 154]. Note also that in the *nilpotent* case there is a cohomological interpretation of the conditions in Theorem 1 (see [9, Proposition 4]).

In particular, we obtain from Theorem 1:

Corollary 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then all simple $u(L,\chi)$-modules with the same eigenvalue function have the same dimension.* □

Consider for every $\chi \in L^*$ and every $\tau \in \Pi(L,\chi)$ the set of isomorphism classes of simple $u(L,\chi)$-modules with eigenvalue function τ:

$$\mathrm{Irr}_\tau(L,\chi) := \{[S] \in \mathrm{Irr}(L,\chi) \mid (t)_S = \tau(t) \cdot \mathrm{id}_S \;\; \forall\, t \in T_p(L)\}.$$

If we set

$$G_0^L(S) := \{\gamma \in G_0^L \mid F_\gamma \otimes_{\mathbb{F}} S \cong S\},$$

then we obtain the following refinement of [16, Proposition 5.8.8]:

Proposition 1 *Let L be a finite-dimensional restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then the following statements hold:*

(a) *There is a partition*

$$\operatorname{Irr}(L,\chi) = \bigcup_{\tau \in \Pi(L,\chi)} \operatorname{Irr}_\tau(L,\chi).$$

(b) *If $\tau \in \Pi(L,\chi)$, then G_0^L acts on $\operatorname{Irr}_\tau(L,\chi)$ via $\gamma \cdot [S] := [F_\gamma \otimes_{\mathbb{F}} S]$ for $\gamma \in G_0^L$ and $[S] \in \operatorname{Irr}_\tau(L,\chi)$.*

Let S denote a(n arbitrary) simple $u(L,\chi)$-module with eigenvalue function τ.

(c) *Then the mapping*

$$\Phi_0^L(S) : \begin{cases} G_0^L/G_0^L(S) \longrightarrow \operatorname{Irr}_\tau(L,\chi) \\ \gamma + G_0^L(S) \longmapsto [F_\gamma \otimes_{\mathbb{F}} S] \end{cases}$$

is injective.

(d) *If L is supersolvable, then G_0^L acts transitively on $\operatorname{Irr}_\tau(L,\chi)$ for every $\tau \in \Pi(L,\chi)$, i.e., the mapping $\Phi_0^L(S)$ is bijective. In particular, $|\operatorname{Irr}_\tau(L,\chi)|$ is a p-power.*

Proof. (a): Let S be a simple $u(L,\chi)$-module. As already was observed above, we obtain from Schur's lemma the existence of an element $\sigma \in \Pi(L,\chi)$ such that

$$S_{|T(L)} \cong F_\sigma^{\oplus \dim_{\mathbb{F}} S},$$

i.e., $[S] \in \operatorname{Irr}_\sigma(L,\chi)$. Since σ is uniquely determined by S, $\operatorname{Irr}_\tau(L,\chi)$ and $\operatorname{Irr}_\sigma(L,\chi)$ are disjoint if $\tau \neq \sigma$. Finally, [15, Satz 1] yields $\operatorname{Irr}_\tau(L,\chi) \neq \emptyset$ for every $\tau \in \Pi(L,\chi)$.

(b) follows directly from the definitions of G_0^L and $\operatorname{Irr}_\tau(L,\chi)$, and (c) is an immediate consequence of (b).

(d): According to Theorem 1, G_0^L acts transitively on $\operatorname{Irr}_\tau(L,\chi)$ for every $\tau \in \Pi(L,\chi)$, i.e., the mapping $\Phi_0^L(S)$ is surjective for a(ny) simple module S in $\operatorname{Irr}_\tau(L,\chi)$. Hence we have

$$|\operatorname{Irr}_\tau(L,\chi)| = \frac{|G_0^L|}{|G_0^L(S)|}.$$

Since G_0^L (as a subgroup of G^L) is a finite p-group, the second part of (d) also follows. □

Remark. Note that in the special case $T_p(L) = 0$ one obtains $\operatorname{Irr}(L,\chi) = \operatorname{Irr}_0(L,\chi)$. Since $T_p(L) = 0$ if and only if $C(L)$ is p-nilpotent (see [9, Theorem 4]), Proposition 1(d) generalizes [16, Proposition 5.8.8(4)].

Define

$$u_\tau(L,\chi) := u(L,\chi)/u(L,\chi)\{t - \tau(t) \cdot 1 \mid t \in T_p(L)\}$$

for every $\tau \in \Pi(L,\chi)$. These algebras were already implicitly considered by the author in [9, Example 4] and in general by R. Farnsteiner in [4, §2]. It is clear from the definition of $u_\tau(L,\chi)$ that the set of isomorphism classes of simple $u_\tau(L,\chi)$-modules is in bijection with $\mathrm{Irr}_\tau(L,\chi)$.

Since $u(L,0)T_p(L)$ is a Hopf ideal of $u(L,0)$, $u_0(L,0)$ is a Hopf algebra. (In fact, $u(L,0)\{t-\tau(t)\cdot 1 \mid t \in T_p(L)\}$ is a Hopf ideal if and only if $\tau = 0$.) Moreover, it follows from the universal property of restricted universal enveloping algebras that

$$u_0(L,0) \cong u(L/T_p(L),0).$$

Recall that a Lie algebra L is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_L(x)) = 0$ for every $x \in L$. The next result tells us that the representation theory of $u(L,\chi)$ reduces to the representation theory of $u_\tau(L,\chi)$ for $\tau \in \Pi(L,\chi)$. Furthermore, the latter are also always Frobenius algebras which all have the same dimension.

Theorem 2 *Let L be a finite-dimensional restricted Lie algebra over an arbitrary field $\mathbb{F}$ and let $\chi \in L^*$. Then there is a decomposition*

$$u(L,\chi) \cong \bigoplus_{\tau\in\Pi(L,\chi)} u_\tau(L,\chi)$$

and the following statements hold for every $\tau \in \Pi(L,\chi)$:

(a) $\dim_{\mathbb{F}} u_\tau(L,\chi) = p^{\dim_{\mathbb{F}} L/T_p(L)}$.

(b) $u_\tau(L,\chi)$ *is a Frobenius algebra.*

(c) *If L is unimodular, then $u_\tau(L,\chi)$ is symmetric.*

Proof. (a): Set $T := T_p(L)$ and consider the eigenspaces

$$E_\tau := \{u \in u(L,\chi) \mid tu = \tau(t)\cdot u \quad \forall\, t \in T\}$$

for $\tau \in \Pi(L,\chi)$. Since T is abelian and acts semisimply on $u(L,\chi)$, a standard result in linear algebra shows that T is simultaneously diagonalizable on $u(L,\chi)$, i.e., there exists a subset Π of $\Pi(L,\chi)$ such that

$$u(L,\chi) = \bigoplus_{\tau\in\Pi} E_\tau.$$

In particular, there are elements $e_\tau \in E_\tau$ with

$$1 = \sum_{\tau\in\Pi} e_\tau.$$

Because of $T \subseteq C(u(L,\chi))$, we obtain that

$$E_\tau E_{\tau'} \subseteq E_\tau \cap E_{\tau'} = 0$$

for $\tau \neq \tau'$, and it follows from a straightforward computation that

$$ue_\tau = u = e_\tau u$$

for every $u \in E_\tau$ and every $\tau \in \Pi$. In particular, $e_\tau = e_\tau^2$ and $E_\tau = u(L,\chi)e_\tau = e_\tau u(L,\chi)$ is a two-sided central ideal for every $\tau \in \Pi$. Hence $\{e_\tau \mid \tau \in \Pi\}$ is a complete set of pairwise orthogonal central idempotents of $u(L,\chi)$.

Since $\Pi = \Pi(L,\chi)$ will be proved below, it remains to establish that $u_\tau(L,\chi) \cong E_\tau$ for every $\tau \in \Pi(L,\chi)$. Let $\tau \in \Pi(L,\chi)$ and consider the canonical projection η_τ from $u(L,\chi)$ onto E_τ. It is clear from the properties of the e_τ's that η_τ is an $\mathbb{F}$-algebra epimorphism and again standard linear algebra shows that

$$\mathrm{Ker}(\eta_\tau) = \sum_{t \in T} \mathrm{Im}[(t)_{u(L,\chi)} - \tau(t) \cdot \mathrm{id}_{u(L,\chi)}] = u(L,\chi)\{t - \tau(t) \cdot 1 \mid t \in T_p(L)\}.$$

(b): Let $\{\overline{x}_1, \ldots, \overline{x}_n\}$ be a basis of the L/T. Then it follows from the PBW theorem for reduced universal enveloping algebras (see [16, Theorem 5.3.1]) that

$$\{\overline{x}_1^{a_1}, \ldots, \overline{x}_n^{a_n} \mid 0 \leq a_j \leq p-1 \text{ for every } 1 \leq j \leq n\}$$

is a generating set of $u_\tau(L,\chi)$, i.e.,

$$\dim_{\mathbb{F}} u_\tau(L,\chi) \leq p^{\dim_{\mathbb{F}} L/T}$$

for every $\tau \in \Pi(L,\chi)$. From this and Lemma 2 below it is now clear that $\Pi = \Pi(L,\chi)$ and $\dim_{\mathbb{F}} u_\tau(L,\chi) = p^{\dim_{\mathbb{F}} L/T}$ for every $\tau \in \Pi(L,\chi)$.

(c): Since by the proof of (a) we have $u_\tau(L,\chi) \cong E_\tau$, it is enough to show that E_τ is a Frobenius algebra for every $\tau \in \Pi(L,\chi)$. According to [16, Corollary 5.4.3], $u(L,\chi)$ is a Frobenius algebra, i.e., there exists a linear form φ on $u(L,\chi)$ such that $\varphi(xu(L,\chi)) = 0$ implies $x = 0$. Again using the orthogonality of the E_τ's, we see that the same conclusion holds for the restriction of φ to E_τ.

(d): If L is unimodular, then it follows from [16, Theorem 5.4.2(2) and Corollary 5.4.3] that $u(L,\chi)$ is symmetric, and (d) can be proved similarly to (c). □

Remark. Note that

$$E_\tau = u(L,\chi)e_\tau \cong \mathrm{Ind}_{T_p(L)}^{L}(\mathbb{F} \cdot e_\tau, \chi)$$

for every $\tau \in \Pi(L,\chi)$ (see [14, p. 192]). Moreover, if the ground field is algebraically closed, then the e_τ's can be given explicitly, namely, if $\{t_1, \ldots, t_r\}$ is a toral basis of $T_p(L)$ (which exists by Jacobson's construction [16, Theorem 2.3.6(1)]), then set

$$e_\tau = \prod_{i=1}^{r} [1 - (t_i - \tau(t_i) \cdot 1)]^{p-1}$$

for every $\tau \in \Pi(L,\chi)$ (cf. [14, p. 192] and the proof of [8, Theorem 1]).

Since the block ideals of $u(L,\chi)$ are its *indecomposable* (two-sided) ideals, it is an immediate consequence of (the proof of) Theorem 2 that $u_\tau(L,\chi)$ is a direct sum of certain block ideals of $u(L,\chi)$ (see also [4, Theorem 2.15(2)]).

We conclude this section by illustrating Theorem 2 in the case of *nilpotent* and *supersolvable* restricted Lie algebras. It follows from the remark after Theorem 1 that in the nilpotent case we have

$$|\mathrm{Irr}_\tau(L,\chi)| = 1$$

for every $\tau \in \Pi(L,\chi)$ and thus

Corollary 2 (cf. [8, Theorem 1(a)] and [4, Corollary 2.6(4)]) *Let L be a finite-dimensional nilpotent restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then every block ideal of $u(L,\chi)$ is isomorphic to $u_\tau(L,\chi)$ for some $\tau \in \Pi(L,\chi)$.* □

The block of $u(L, 0)$ which contains the one-dimensional trivial L-module is called the *principal block* of L and will be denoted by $B_0(L)$.

Corollary 3 (cf. [3, Corollary 4.5]) *If L is a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$, then the principal block ideal $B_0(L)$ is isomorphic to $u_0(L, 0)$.*

Proof. By virtue of [9, Theorem 5], $B_0(L)$ is isomorphic to a direct summand of $u_0(L, 0)$ and $\mathrm{Irr}_0(L, 0) = \mathrm{Irr}(B_0(L))$ which implies the assertion. □

Remark. Corollary 3 can immediately be generalized to the situation of blocks with a (and thus only) one-dimensional simple module(s) which always have a *transitive* group action (see [7] and [4, Proposition 2.2])).

§2. In this section we decompose a certain induced projective $u(L, \chi)$-module of a finite-dimensional supersolvable restricted Lie algebra L into projective indecomposable modules and use this in order to derive a formula for the number of isomorphism classes of simple $u(L, \chi)$-modules with a fixed eigenvalue function. As a consequence we will also obtain in §3 an upper bound on the number of isomorphism classes of all simple $u(L, \chi)$-modules.

Recall that a projective module $P(M)$ is a *projective cover* of a module M if there exists a module epimorphism π_M from $P(M)$ onto M such that the kernel of π_M is contained in the radical of $P(M)$. It is well-known that projective covers of finite-dimensional modules over finite-dimensional associative algebras always exist and are again finite-dimensional. Moreover, the *projective indecomposable* modules of a finite-dimensional associative algebra are isomorphic to the projective covers of the simple modules.

For the proof of our next result we employ the following universal property of the pair $(P(M), \pi_M)$ which is an immediate consequence of Nakayama's lemma:

(PC) If P is a projective module and π is a module epimorphism from P onto M, then every module homomorphism η from P into $P(M)$ with $\pi_M \circ \eta = \pi$ is an epimorphism.

Remark. Since P is projective and π_M is an epimorphism, there *always* exists a module homomorphism η from P into $P(M)$ such that $\pi_M \circ \eta = \pi$. In particular, it follows from (PC) that projective covers are unique up to isomorphism.

Lemma 1 *Let L be a finite-dimensional restricted Lie algebra over an arbitrary field $\mathbb{F}$ and let $\chi, \chi' \in L^*$. If S is a simple $u(L, \chi)$-module and F is a one-dimensional $u(L, \chi')$-module, then*

$$P(F \otimes_{\mathbb{F}} S) \cong F \otimes_{\mathbb{F}} P(S).$$

Proof. Set $P := F \otimes_{\mathbb{F}} P(S)$. According to [6, Lemma 2.3], P is a projective $u(L, \chi + \chi')$-module. If π_S denotes the epimorphism from $P(S)$ onto S, then $\mathrm{id}_F \otimes \pi_S$ is an epimorphism from P onto $F \otimes_{\mathbb{F}} S$. We conclude from (PC) that $P(F \otimes_{\mathbb{F}} S)$ is a direct summand of P. Since S is simple, $P(S)$ is indecomposable. But F is one-dimensional and by tensoring P with the dual F^* of F we see that P is also indecomposable which implies that P and $P(F \otimes_{\mathbb{F}} S)$ are isomorphic. □

Remark. Lemma 1 was already proved in [5, Satz II.3.5a)] for the special case that S is the one-dimensional trivial L-module. Note that in the light of Lemma 1 the statements in Theorem 1 are also equivalent to $P(N) \cong F_\gamma \otimes_{\mathbb{F}} P(M)$ (see [11, Theorem 2]).

The next result generalizes [8, Theorem 1(b)] from *nilpotent* to *supersolvable* restricted Lie algebras. Note that the proof of Theorem 3 is very similar to the proof of [10, Theorem 5.2].

Theorem 3 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$, let $\chi \in L^*$, and let S be a simple $u(L,\chi)$-module with eigenvalue function σ. If $\Gamma_0^L(S)$ denotes a complete set of representatives of the (left) cosets of $G_0^L(S)$ in G_0^L, then*

$$\operatorname{Ind}_{T_p(L)}^L(F_\sigma,\chi) \cong \bigoplus_{\gamma\in\Gamma_0^L(S)} [F_\gamma \otimes_{\mathbb{F}} P(S)]^{\oplus \dim_{\mathbb{F}} S}.$$

Proof. According to [10, Lemma 3.2], F_σ is a projective $u(T_p(L),\chi_{|T_p(L)})$-module, and we conclude from the additivity of the induction functor that

$$P := \operatorname{Ind}_{T_p(L)}^L(F_\sigma,\chi)$$

is projective. Since $\{P(X) \mid X \in \operatorname{Irr}(L,\chi)\}$ is a full set of representatives of isomorphism classes of projective indecomposable $u(L,\chi)$-modules, there exist non-negative integers m_X such that

$$P \cong \bigoplus_{X\in\operatorname{Irr}(L,\chi)} P(X)^{\oplus m_X}.$$

If X has eigenvalue function ξ, then we conclude from [1, Lemma 1.7.5] and Frobenius reciprocity in conjunction with Theorem 1 that

$$\begin{aligned} m_X &= \dim_{\mathbb{F}} \operatorname{Hom}_L(P,X) \\ &= \dim_{\mathbb{F}} \operatorname{Hom}_{T_p(L)}(F_\sigma, X_{|T_p(L)}) \\ &= \dim_{\mathbb{F}} \operatorname{Hom}_{T_p(L)}(F_\sigma, F_\xi)^{\oplus \dim_{\mathbb{F}} X} \\ &= \begin{cases} \dim_{\mathbb{F}} S & \text{if } X \cong F_\gamma \otimes_{\mathbb{F}} S \text{ for some } \gamma \in G_0^L \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Finally, this together with Lemma 1 implies that

$$P \cong \bigoplus_{\gamma\in\Gamma_0^L(S)} [F_\gamma \otimes_{\mathbb{F}} P(S)]^{\oplus \dim_{\mathbb{F}} S}. \quad \square$$

§3. Let L be a finite-dimensional restricted Lie algebra and let $\lambda \in L^*$. A p-subalgebra K of L is called *polarization* of λ if K is a maximal isotropic subspace with respect to the alternating form b_λ on L defined by $b_\lambda(x,y) := \lambda([x,y])$ for every $x,y \in L$. If $\chi \in L^*$, then a polarization of λ is called χ-*admissible* if $\lambda(y)^p - \lambda(y^{[p]}) = \chi(y)^p$ for every $y \in K$. Since $T_p(L)$ is central, $T_p(L)$ is contained in every polarization K, and thus $T_p(L)$ is contained in every maximal torus of K.

If L is supersolvable over an algebraically closed field, then B. Yu. Veisfeiler and V. G. Kac [17, Theorem 1] have shown that for every simple $u(L,\chi)$-module S there exists $\lambda \in L^*$ and a χ-admissible polarization K of λ with

$$(*) \qquad S \cong \operatorname{Ind}_K^L(F_\lambda,\chi).$$

This and a formula for the dimension of their projective covers (see [4, Theorem 2.9(2)] or [11, Corollary 1]) in conjunction with Proposition 1 and Theorem 3 can be used to derive a formula for the number of isomorphism classes of simple $u(L,\chi)$-modules with the same eigenvalue function.

Proposition 2 (cf. [4, Theorem 2.15(2)]) *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then for every $\tau \in \Pi(L, \chi)$ there exists a χ-admissible polarization K such that*

$$|\mathrm{Irr}_\tau(L, \chi)| = p^{\dim_{\mathbb{F}} T_{\max}(K)/T_p(L)}$$

holds for a(ny) maximal torus $T_{\max}(K)$ of K.

Proof. According to (the proof of) Proposition 1(d), G_0^L acts transitively on $\mathrm{Irr}_\tau(L, \chi)$ for every $\tau \in \Pi(L, \chi)$, i.e., for a(ny) simple module S in $\mathrm{Irr}_\tau(L, \chi)$ we have

$$|\mathrm{Irr}_\tau(L, \chi)| = \frac{|G_0^L|}{|G_0^L(S)|} = |\Gamma_0^L(S)|.$$

Then an application of Theorem 3 in conjunction with $(*)$ and the dimension formula for projective indecomposable modules mentioned above yields:

$$|\mathrm{Irr}_\tau(L, \chi)| = |\Gamma_0^L(S)| = \frac{p^{\dim_{\mathbb{F}} L/T_p(L)}}{p^{\dim_{\mathbb{F}} L/T_{\max}(K)}} = p^{\dim_{\mathbb{F}} T_{\max}(K)/T_p(L)}. \quad \square$$

Remark. Note that Corollary 3 in conjunction with Proposition 2 and [9, Theorem 1] (or the fact that $L/T_p(L)$ is a semidirect product of a torus and a p-nilpotent restricted Lie algebra) gives another proof of [9, Theorem 6].

In order to derive from Proposition 1(a) and Proposition 2 lower and upper bounds for the number of isomorphism classes of simple $u(L, \chi)$-modules, we need to determine the number of elements in $\Pi(L, \chi)$ (cf. [14, p. 191]).

Lemma 2 *Let L be a finite-dimensional restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then*

$$|\Pi(L, \chi)| = p^{\dim_{\mathbb{F}} T_p(L)}.$$

Proof. According to [16, Theorem 2.3.6(1)], we can choose a toral basis $\mathcal{T} := \{t_1, \ldots, t_r\}$ of $T_p(L)$. Let $\{\tau_1, \ldots, \tau_r\}$ denote the dual basis of $\mathcal{T}$ and choose for every $1 \le i \le r$ a root δ_i of the polynomial $X^p - X - \chi(t_i)^p$. If $\tau \in \Pi(L, \chi)$, then a straightforward computation shows that the linear form $\tilde{\tau} \in T_p(L)^*$ defined by $\tilde{\tau}(t_i) := \tau(t_i) - \delta_i$ for every $1 \le i \le r$ is contained in $\Pi(L, 0)$. Obviously, this defines a bijection from $\Pi(L, \chi)$ onto $\Pi(L, 0)$. But

$$\Pi(L, 0) = \bigoplus_{i=1}^{r} \mathbb{F}_p \tau_i,$$

and therefore we obtain that $|\Pi(L, \chi)| = |\Pi(L, 0)| = p^r$. $\square$

Remark. Note that Lemma 2 in conjunction with the observations at the end of §1 gives another proof of [9, Lemma 3(b)].

By combining Proposition 1(a), Proposition 2, and Lemma 2, we finally can derive the following lower and upper bounds for the number of isomorphism classes of simple $u(L, \chi)$-modules which generalize [15, Satz 6] from *nilpotent* to *supersolvable* restricted Lie algebras.

Theorem 4 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. If $T_{\max}(L)$ denotes a(n arbitrary) maximal torus of L, then*

$$p^{\dim_{\mathbb{F}} T_p(L)} \leq |\mathrm{Irr}(L,\chi)| \leq p^{\dim_{\mathbb{F}} T_{\max}(L)}$$

holds.

Proof. By virtue of [15, Satz 1], $\mathrm{Irr}_\tau(L,\chi) \neq \emptyset$ for each $\tau \in \Pi(L,\chi)$, and thus the lower bound is an immediate consequence of Proposition 1(a) and Lemma 2. Finally, the upper bound follows from Proposition 1(a) in conjunction with Lemma 2 and Proposition 2. □

Let us now consider the important special case of a strongly solvable restricted Lie algebra over an algebraically closed ground field. Recall that a restricted Lie algebra L is called *strongly solvable* if $[L,L]$ is p-nilpotent. Then we have the following analogue of Proposition 2.

Theorem 5 *Let L be a finite-dimensional strongly solvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and let $\chi \in L^*$. Then there exists a χ-admissible polarization K such that*

$$|\mathrm{Irr}(L,\chi)| = p^{\dim_{\mathbb{F}} T_{\max}(K)}$$

holds for a(ny) maximal torus $T_{\max}(K)$ of K. In particular, $|\mathrm{Irr}(L,\chi)|$ is always a p-power.

Proof. Since L is strongly solvable and $\mathbb{F}$ is algebraically closed, *every* simple restricted L-module is one-dimensional, and the argument in the proof of the implication (a)$\Longrightarrow$(c) in Theorem 1 shows that G^L acts transitively on $\mathrm{Irr}(L,\chi)$ for every $\chi \in L^*$ (see also [13, §8.4]). If S is an arbitrary simple $u(L,\chi)$-module, then it follows from [11, Corollary 4 and the remark after Corollary 3] that

$$|\mathrm{Irr}(L,\chi)| = \frac{|G^L|}{|G^L(S)|} = p^{\dim_{\mathbb{F}} T_{\max}(K)}$$

for a(ny) maximal torus $T_{\max}(K)$ of a χ-admissible polarization K of some $\lambda \in L^*$ such that $S \cong \mathrm{Ind}_K^L(F_\lambda,\chi)$. □

The concluding example shows that in general $|\mathrm{Irr}(L,\chi)|$ is *not* a p-power. In particular, this implies that the formula for $|\mathrm{Irr}(L,\chi)|$ in [2, Corollary 1 of Theorem 4, p. 215] cannot be valid. Moreover, we can see that neither the lower nor the upper bound in Theorem 4 is the best possible.

Example (cf. [10, Example (ii) in §2]) Consider the three-dimensional supersolvable restricted Lie algebra

$$L = \mathbb{F}t \oplus \mathbb{F}e \oplus \mathbb{F}z,$$

$$[t,e] = e + z, \quad [z,L] = 0, \qquad t^{[p]} = t, \quad e^{[p]} = 0, \quad z^{[p]} = z,$$

and let χ be an arbitrary linear form on L. Observe that $T_p(L) = \mathbb{F}z$ which implies that τ is uniquely determined by $\tau(z)$ for every $\tau \in \Pi(L,\chi)$.

It is clear from the definition of L that there exists a *one-dimensional* $u_\tau(L,\chi)$-module if and only if $\tau(z) = -\chi(e)$. According to Corollary 1, the latter condition is satisfied if and only if *every* simple $u_\tau(L,\chi)$-module is *one-dimensional*. Hence it follows (from Proposition 2) that in this case $|\mathrm{Irr}_\tau(L,\chi)| = p$.

If $\tau(z) \neq -\chi(e)$, then by virtue of Theorem 2(a) every $u_\tau(L,\chi)$-module is p-dimensional, and thus $|\mathrm{Irr}_\tau(L,\chi)| = 1$.

Since $|\Pi(L,\chi)| = p$, we can conclude that

$$|\mathrm{Irr}(L,\chi)| = p + (p-1) \cdot 1 = 2p - 1.$$

Remark. Note that it is a consequence of [12] that $u_\tau(L,\chi)$ is of finite module type if $\tau(z) = -\chi(e)$, and it can be read off from Theorem 2(a) that $u_\tau(L,\chi)$ is semisimple if $\tau(z) \neq -\chi(e)$.

Acknowledgments

The author would like to thank the Mathematical Institute at the University of Oxford for the hospitality during his visit when parts of this paper were written. In particular, it is a pleasure to thank Karin Erdmann for the productive and pleasant time in Oxford. The author also gratefully acknowledges the financial support by the DFG.

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