

Projective Modules and Blocks of Supersolvable Restricted Lie Algebras

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In this paper we investigate the projective indecomposable modules for reduced universal enveloping algebras of a finite-dimensional supersolvable restricted Lie algebra and thereby generalize the results in [7]. Moreover, we improve several results which are related to [8, Questions 1 & 2].

In the first main result (see Theorem 1) we show how the projective cover of the induced module of a one-dimensional module appears as a direct summand in the induced module of the projective cover of the one-dimensional module. Since this result becomes meaningless for one-dimensional modules, we also re-derive a result due to R. Farnsteiner on projective covers of one-dimensional modules by reducing it to the strongly solvable case. The latter is already contained in the author's doctoral dissertation (see [5, Satz II.3.2]), and our proof shows that the general case is an easy consequence of the special case.

In the second main result (see Theorem 2) we improve on [8, Theorem 3] by restricting the group acting on the simple modules as well as by considering also projective indecomposable modules and their radicals. In particular, we obtain that the Loewy lengths of projective indecomposable modules belonging to the same block coincide (see Corollary 2) and that the number of the isomorphism classes of simple modules in an arbitrary block of a finite-dimensional supersolvable restricted Lie algebra is a p -power. Moreover, we also can improve the upper bound from [8, Corollary 5].

In the third main result (see Theorem 3) we generalize [7, Theorem 2] and an unpublished result of J. C. Jantzen to the supersolvable case. This enables us to compute the order of certain stabilizers (see Corollary 4).

Finally, we determine the dimensions of the block ideals and their Jacobson radicals up to the number of isomorphism classes of simple modules. As a consequence we obtain an upper bound for the dimensions of the block ideals (see Proposition 2) and a characterization of semisimple blocks similarly to the nilpotent case (see Corollary 6). Moreover, we also deduce that reduced universal enveloping algebras of strongly solvable restricted Lie algebras are always blockstable (see Proposition 4).

In the following let \mathbb{F} always be a field of prime characteristic p . For more notation and fundamental facts from the representation theory of restricted Lie algebras we refer the reader to [15] and for block theory to [12].

A Lie algebra L is called *supersolvable* if there exists a (descending) chain

$$L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$$

of ideals L_j in L such that the factor algebras L_j/L_{j+1} are one-dimensional for every $0 \leq j \leq n-1$.

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$n - 1$. It is well-known that subalgebras and factor algebras of supersolvable Lie algebras are again supersolvable. Furthermore, every supersolvable Lie algebra is *solvable*. Conversely, by Lie's theorem every finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic zero is supersolvable but this does not hold in prime characteristic. In order to overcome this difficulty one introduces the following concept. A restricted Lie algebra L is called *strongly solvable* if $[L, L]$ is p -nilpotent. Hence every finite-dimensional *restricted* simple module over a finite-dimensional strongly solvable Lie algebra is *one-dimensional*. Moreover, it turns out that supersolvable restricted Lie algebras are similarly related to strongly solvable ones as nilpotent restricted Lie algebras are related to p -nilpotent ones.

Recall that a projective module $P(M)$ is a *projective cover* of a module M if there exists a module epimorphism π_M from $P(M)$ onto M such that the kernel of π_M is contained in the radical of $P(M)$. It is well-known that projective covers of finite-dimensional modules over finite-dimensional associative algebras always exist and are again finite-dimensional. Moreover, the *projective indecomposable* modules of a finite-dimensional associative algebra are isomorphic to the projective covers of the simple modules.

In the sequel we will need the following universal property of the pair $(P(M), \pi_M)$ which is an immediate consequence of Nakayama's lemma:

(PC) If P is a projective module and π is a module epimorphism from P onto M , then every module homomorphism η from P into $P(M)$ with $\pi_M \circ \eta = \pi$ is an epimorphism.

Remark. Since P is projective and π_M is an epimorphism, there *always* exists a module homomorphism η from P into $P(M)$ such that $\pi_M \circ \eta = \pi$. In particular, it follows from (PC) that projective covers are unique up to isomorphism.

The block of L which contains the one-dimensional trivial module is called the *principal block* of L and will be denoted by $B_0(L)$. Observe that if $C(L)$ denotes the *center* of L , then

$$T_p(L) := \{x \in C(L) \mid x \text{ is semisimple} \}$$

is the largest toral ideal of L . We will need the following generalization of [8, Proposition 3]:

Lemma 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an arbitrary field \mathbb{F} . If S is a simple module belonging to the principal block of L , then $T_p(L) \cdot P_L(S) = 0$.*

Proof. Since $P_L(S)$ is indecomposable and S belongs to the principal block of L , all its composition factors belong to the principal block of L . Hence by [8, Proposition 3], $T_p(L)$ acts nilpotently and thus trivially on $P_L(S)$. \square

For the convenience of the reader we also include a new proof of the next result which in this generality is due to R. Farnsteiner (see [2, Corollary 4.5] and [3, Proposition 2.2]).

Proposition 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} with maximal torus $T_{\max}(L)$ and let $\chi \in L^*$. If S is a one-dimensional $u(L, \chi)$ -module, then*

$$P_L(S) \cong \text{Ind}_{T_{\max}(L)}^L(S, \chi)$$

holds.

Proof. Set $T := T_{\max}(L)$. Suppose for a moment that $L = T \oplus \text{Rad}_p(L)$ is strongly solvable and that S is the one-dimensional trivial (restricted) L -module \mathbb{F} . By virtue of the main result of [11], \mathbb{F} is a projective $u(T, 0)$ -module, and it follows from the additivity of the induction functor that $P := \text{Ind}_T^L(\mathbb{F}, 0)$ is a projective $u(L, 0)$ -module. According to [15, Theorem 5.6.3], there exists an L -module epimorphism from P onto \mathbb{F} . Then we obtain from (PC) that $P_L(\mathbb{F})$ is a homomorphic image (in fact, a direct summand) of P . As a consequence of the PBW theorem for restricted universal enveloping algebras (cf. [15, Theorem 2.5.1]), $u(L, 0)$ is a free $u(\text{Rad}_p(L), 0)$ -module, and therefore $P_L(\mathbb{F})|_{\text{Rad}_p(L)}$ is a projective $u(\text{Rad}_p(L), 0)$ -module. Since restricted universal enveloping algebras of finite-dimensional p -nilpotent restricted Lie algebras are local, we can conclude that $P_L(\mathbb{F})|_{\text{Rad}_p(L)}$ is free (see [6, Proposition 5.1]), and thus we have

$$p^{\dim_{\mathbb{F}} L/T} \leq \dim_{\mathbb{F}} P_L(\mathbb{F}) \leq \dim_{\mathbb{F}} P = p^{\dim_{\mathbb{F}} L/T},$$

i.e., $P_L(\mathbb{F}) \cong P$.

Next, we prove the assertion in the case that L is supersolvable and S is trivial. Because the ground field is algebraically closed, $\bar{L} := L/T_p(L)$ is strongly solvable with maximal torus $\bar{T} := T/T_p(L)$ (cf. the argument in the proof of [8, Theorem 6]). Hence by the above we obtain $P_{\bar{L}}(\mathbb{F}) \cong \text{Ind}_{\bar{T}}^{\bar{L}}(\mathbb{F}, 0)$, i.e.,

$$\dim_{\mathbb{F}} P_{\bar{L}}(\mathbb{F}) = p^{\dim_{\mathbb{F}} \bar{L}/\bar{T}} = p^{\dim_{\mathbb{F}} L/T}.$$

In view of Lemma 1, we have $T_p(L) \cdot P_L(\mathbb{F}) = 0$ which implies that $P_L(\mathbb{F})$ is a projective $u(\bar{L}, 0)$ -module. Then as in the first part of the proof it follows from (PC) that $P_{\bar{L}}(\mathbb{F})$ is a homomorphic image of $P_L(\mathbb{F})$. In particular, we have

$$(*) \quad \dim_{\mathbb{F}} P_L(\mathbb{F}) \geq \dim_{\mathbb{F}} P_{\bar{L}}(\mathbb{F}) = p^{\dim_{\mathbb{F}} L/T}.$$

If S is an arbitrary one-dimensional $u(L, \chi)$ -module, then [10, Lemma 1] in conjunction with (*) shows that

$$(**) \quad \dim_{\mathbb{F}} P_L(S) = \dim_{\mathbb{F}} P_L(\mathbb{F}) \geq p^{\dim_{\mathbb{F}} L/T}.$$

By the same argument as in the first part of the proof, $P_L(S)$ is a homomorphic image of $\text{Ind}_T^L(S, \chi)$ which in conjunction with (**) finally yields $P_L(S) \cong \text{Ind}_T^L(S, \chi)$. \square

Remark. Note that it is a consequence of the proof of Proposition 1 that $P_{L/T_p(L)}(S) \cong P_L(S)$ (as restricted $L/T_p(L)$ -modules) for every simple module S belonging to the principal block of L .

Consider the (commutative p -)subgroup

$$G^L := \{\gamma \in L^* \mid \gamma([L, L]) = 0, \gamma(x^{[p]}) = \gamma(x)^p \quad \forall x \in L\}$$

of the (additive) group L^* (cf. [15, p. 242]). As a consequence of the Jordan-Chevalley-Seligman-Schue decomposition, G^L is finite (see [15, Proposition 5.8.8(1)]). For every $\gamma \in G^L$ the one-dimensional vector space $F_\gamma := \mathbb{F} \cdot 1_\gamma$ is a restricted L -module via $x \cdot 1_\gamma := \gamma(x) \cdot 1_\gamma$, and conversely, every one-dimensional restricted L -module occurs in this way.

It is obvious that

$$G_0^L := \{\gamma \in G^L \mid \gamma|_{T_p(L)} = 0\}$$

is a subgroup of G^L . Note that by virtue of [8, Theorem 5] for *supersolvable* restricted Lie algebras this definition of G_0^L coincides with the definition in [8, p. 411]. If S is a simple L -module, then set

$$G_0^L(S) := \{\gamma \in G_0^L \mid F_\gamma \otimes_{\mathbb{F}} S \cong S\}$$

and denote by $\Gamma_0^L(S)$ a complete set of representatives of the (one-sided) cosets of $G_0^L(S)$ in G_0^L .

Let S be a simple L -module and assume that the ground field \mathbb{F} is algebraically closed. Since $T_p(L) \subseteq C(L)$ and S is simple, Schur's lemma implies that for every $t \in T_p(L)$ there exists an element $\sigma(t) \in \mathbb{F}$ with $(t)_S = \sigma(t) \cdot \text{id}_S$, i.e.,

$$S_{|T_p(L)} \cong F_\sigma^{\oplus \dim_{\mathbb{F}} S}.$$

Since σ is uniquely determined by S , we call σ the *eigenvalue function* of S (cf. [10, §1]).

Let L be a finite-dimensional restricted Lie algebra and let $\lambda \in L^*$. A p -subalgebra K of L is called *polarization* of λ if K is a maximal isotropic subspace with respect to the alternating form b_λ on L defined by $b_\lambda(x, y) := \lambda([x, y])$ for every $x, y \in L$. If $\chi \in L^*$, then a polarization of λ is called χ -*admissible* if $\lambda(y)^p - \lambda(y^{[p]}) = \chi(y)^p$ for every $y \in K$. Since $T_p(L)$ is central, $T_p(L)$ is contained in every polarization K , and thus $T_p(L)$ is contained in every maximal torus of K . If L is supersolvable over an algebraically closed field, then B. Yu. Veisfeiler and V. G. Kac have shown in [16, Theorem 1] that for every simple $u(L, \chi)$ -module S there exists $\lambda \in L^*$ and a χ -admissible polarization K of λ with $S \cong \text{Ind}_K^L(F_\lambda, \chi)$, and conversely that these modules are always simple, i.e., they gave a complete description of the set $\text{Irr}(L, \chi)$ of isomorphism classes of simple $u(L, \chi)$ -modules.

Finally, if M is an L -module and H is a subalgebra of L , then set

$$M_\theta(H) := \{m \in M \mid h \cdot m = \theta(h) \cdot m \quad \forall h \in H\}$$

for any $\theta \in H^*$. Now we can formulate the first main result of this paper which should be viewed as a description of the projective indecomposable $u(L, \chi)$ -modules in terms of projective indecomposable modules for certain natural subalgebras.

Theorem 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} , let $\chi \in L^*$, and let S be a simple $u(L, \chi)$ -module. If K is a χ -admissible polarization of $\lambda \in L^*$ such that $S \cong \text{Ind}_K^L(F_\lambda, \chi)$, then*

$$\text{Ind}_K^L(P_K(F_\lambda), \chi) \cong \bigoplus_{\gamma \in \Gamma_0^L(S)} [F_\gamma \otimes_{\mathbb{F}} P_L(S)]^{\oplus \dim_{\mathbb{F}} S_{\lambda - \gamma}(T_{\max}(K))}$$

for any maximal torus $T_{\max}(K)$ of K .

Proof. It follows from the additivity of induction that

$$P := \text{Ind}_K^L(P_K(F_\lambda), \chi)$$

is projective. Since $\{P_L(X) \mid X \in \text{Irr}(L, \chi)\}$ is a full set of representatives of isomorphism classes of projective indecomposable $u(L, \chi)$ -modules, there exist non-negative integers m_X such that

$$P \cong \bigoplus_{X \in \text{Irr}(L, \chi)} P_L(X)^{\oplus m_X}.$$

Then we conclude from [1, Lemma 1.7.5] and Proposition 1 in conjunction with a twofold application of Frobenius reciprocity that

$$\begin{aligned} m_X &= \dim_{\mathbb{F}} \text{Hom}_L(P, X) \\ &= \dim_{\mathbb{F}} \text{Hom}_K(P_K(F_\lambda), X_{|K}) = \dim_{\mathbb{F}} \text{Hom}_K(\text{Ind}_{T_{\max}(K)}^K(F_\lambda), \chi_{|K}), X_{|K}) \\ &= \dim_{\mathbb{F}} \text{Hom}_{T_{\max}(K)}(F_\lambda, X_{|T_{\max}(K)}) = \dim_{\mathbb{F}} X_\lambda(T_{\max}(K)) \end{aligned}$$

for every simple $u(L, \chi)$ -module X .

Suppose that $X_\lambda(T_{\max}(K)) \neq 0$. Since $T_p(L) \subseteq T_{\max}(K)$ and X is a simple L -module, $\lambda|_{T_p(L)}$ is the eigenvalue function of X . But obviously S has the same eigenvalue function. Hence by [10, Theorem 1] there exists $\gamma \in G_0^L$ such that $X \cong F_\gamma \otimes_{\mathbb{F}} S$ (as L -modules), and in particular, $\dim_{\mathbb{F}} X_\lambda(T_{\max}(K)) = \dim_{\mathbb{F}} S_{\lambda-\gamma}(T_{\max}(K))$. Finally, this together with [10, Lemma 1] yields the desired isomorphism. \square

Remark. Note that $\text{Ind}_K^L(P_K(F_\lambda), \chi)$ is *isotypic* if and only if $G_0^L = G_0^L(S)$, and

$$\text{Ind}_K^L(P_K(F_\lambda), \chi) \cong \bigoplus_{\gamma \in \Gamma_0^L(S)} [F_\gamma \otimes_{\mathbb{F}} P(S)]^{\oplus \dim_{\mathbb{F}} S}$$

if $T_{\max}(K) = T_p(L)$ (compare also with [10, Theorem 3]). The latter condition holds if K is an ideal of L or if K acts nilpotently on L (or in view of [3, Lemma 2.8] if K is a nilpotent V -polarization). Moreover, because of Fong's dimension formula for projective indecomposable modules (cf. [14, Satz 2.3]) the analogue of Theorem 1 in the modular representation theory of finite groups is much simpler (see [17, Proposition 2.7b]).

As a consequence of Theorem 1 we obtain a dimension formula for the projective indecomposable modules (see also [3, Corollary 2.10(b)]) and a relation between $P_L(S)$ and $P_K(F_\lambda)$ via restriction (cf. [3, Theorem 2.9(3)]).

Corollary 1 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} , let $\chi \in L^*$, and let S be a simple $u(L, \chi)$ -module. If K is a χ -admissible polarization of $\lambda \in L^*$ such that $S \cong \text{Coind}_K^L(F_\lambda, \chi)$, then $P_L(S)|_K \cong P_K(F_\lambda)$. In particular,*

$$\dim_{\mathbb{F}} P_L(S) = p^{\dim_{\mathbb{F}} K/T_{\max}(K)}$$

for any maximal torus $T_{\max}(K)$ of K .

Proof. Set $T := T_{\max}(K)$. In view of Proposition 1 and [15, Proposition 5.6.2], the dimension formula for $P_L(S)$ can be read off from the isomorphism in Theorem 1 if

$$\sum_{\gamma \in \Gamma_0^L(S)} \dim_{\mathbb{F}} S_{\lambda-\gamma}(T) = \dim_{\mathbb{F}} S.$$

Let $\theta \in T^*$ such that $S_\theta(T) \neq 0$. Then we have to show that $\vartheta - \lambda \in G_0^L$ for an arbitrary linear extension ϑ of θ to L . (Note that $\gamma|_T = 0$ for every $\gamma \in G_0^L(S)$!) But because of $T_p(L) \subseteq T$, it follows from the proof of Theorem 1 that $\vartheta|_{T_p(L)} = \lambda|_{T_p(L)}$, and thus the claim.

It now remains to prove the first assertion. There exists an K -module epimorphism from $(S$ and thus from) $P_L(S)$ onto F_λ so that we can employ (PC) in conjunction with the dimension formula for $P_L(S)$ to establish an isomorphism between $P_L(S)|_K$ and $P_K(F_\lambda)$. \square

Remark. Note that in view of [6, Corollary 1.2(a)] every simple $u(L, \chi)$ -module can be viewed as a *coinduced* module with the *same* χ -admissible polarization for a *twisted* linear form.

Let $\text{Jac}(A)$ denote the *Jacobson radical* of a finite-dimensional associative algebra A . In order to show that the Loewy lengths of the projective indecomposable $u(L, \chi)$ -modules belonging to the same block coincide, we will need the following more precise version of [10, Lemma 1] which was motivated by [14, Hilfssatz 3.3(b)] from the modular representation theory of p -nilpotent groups.

Lemma 2 *Let L be a finite-dimensional restricted Lie algebra over an arbitrary field \mathbb{F} and let $\chi, \chi' \in L^*$. If M is a simple $u(L, \chi)$ -module and S is a one-dimensional $u(L, \chi')$ -module, then*

$$[\text{Jac}(u(L, \chi))]^n P_L(S \otimes_{\mathbb{F}} M) \cong S \otimes_{\mathbb{F}} [\text{Jac}(u(L, \chi))]^n P_L(M)$$

for every non-negative integer n .

Proof. The case $n = 0$ was already obtained in [10, Lemma 1]. In order to prove the remaining cases, we set $J := \text{Jac}(u(L, \chi))$ as well as $P := P_L(M)$ and proceed by induction on n . In the following we will always identify $P_L(S \otimes_{\mathbb{F}} M)$ and $S \otimes_{\mathbb{F}} P$.

If $n = 1$, then we obtain from the exactness of $\otimes_{\mathbb{F}}$ and the case $n = 0$ that

$$\frac{S \otimes_{\mathbb{F}} P}{S \otimes_{\mathbb{F}} JP} \cong S \otimes_{\mathbb{F}} \frac{P}{JP} \cong S \otimes_{\mathbb{F}} M \cong \frac{P_L(S \otimes_{\mathbb{F}} M)}{JP_L(S \otimes_{\mathbb{F}} M)} \cong \frac{S \otimes_{\mathbb{F}} P}{J(S \otimes_{\mathbb{F}} P)},$$

i.e., $J(S \otimes_{\mathbb{F}} P) \subseteq S \otimes_{\mathbb{F}} JP$ and $\dim_{\mathbb{F}} J(S \otimes_{\mathbb{F}} P) = \dim_{\mathbb{F}} S \otimes_{\mathbb{F}} JP$ whence $J(S \otimes_{\mathbb{F}} P) = S \otimes_{\mathbb{F}} JP$.

If $n > 1$, then suppose that

$$J^{n-1}(S \otimes_{\mathbb{F}} P) \cong S \otimes_{\mathbb{F}} J^{n-1}P$$

already holds and set

$$d_n := \dim_{\mathbb{F}} \frac{J^{n-1}P}{J^n P}$$

as well as

$$e_n := \dim_{\mathbb{F}} \frac{J^{n-1}(S \otimes_{\mathbb{F}} P)}{J^n(S \otimes_{\mathbb{F}} P)}.$$

Observe that

$$\frac{J^{n-1}(S \otimes_{\mathbb{F}} P)}{S \otimes_{\mathbb{F}} J^n P} \cong \frac{S \otimes_{\mathbb{F}} J^{n-1}P}{S \otimes_{\mathbb{F}} J^n P} \cong S \otimes_{\mathbb{F}} \frac{J^{n-1}P}{J^n P}$$

is semisimple because $J^{n-1}P/J^n P$ is semisimple and S is one-dimensional. Hence $J^n(S \otimes_{\mathbb{F}} P) \subseteq S \otimes_{\mathbb{F}} J^n P$ and it follows from the third isomorphism theorem and the induction hypothesis that

$$\frac{J^{n-1}(S \otimes_{\mathbb{F}} P)}{J^n(S \otimes_{\mathbb{F}} P)} / \frac{S \otimes_{\mathbb{F}} J^n P}{J^n(S \otimes_{\mathbb{F}} P)} \cong S \otimes_{\mathbb{F}} \frac{J^{n-1}P}{J^n P},$$

i.e.,

$$e_n - d_n = \dim_{\mathbb{F}} \frac{S \otimes_{\mathbb{F}} J^n P}{J^n(S \otimes_{\mathbb{F}} P)}.$$

Set $X := S^* \otimes_{\mathbb{F}} S \otimes_{\mathbb{F}} J^{n-1}P$ and $Y := S^* \otimes_{\mathbb{F}} J^n(S \otimes_{\mathbb{F}} P)$. Since S is one-dimensional, $X \cong J^{n-1}P$ and furthermore $X \cong S^* \otimes_{\mathbb{F}} J^{n-1}(S \otimes_{\mathbb{F}} P)$ by the induction hypothesis. Therefore

$$X/Y \cong S^* \otimes_{\mathbb{F}} \frac{J^{n-1}(S \otimes_{\mathbb{F}} P)}{J^n(S \otimes_{\mathbb{F}} P)}$$

is semisimple of dimension e_n , i.e., $J^n P \subseteq Y$ and

$$d_n \leq e_n = \dim_{\mathbb{F}} X/Y = \dim_{\mathbb{F}} \frac{X}{J^n P} / \frac{Y}{J^n P} \leq \dim_{\mathbb{F}} \frac{X}{J^n P} = d_n.$$

Hence $e_n = d_n$ and thus $J^n(S \otimes_{\mathbb{F}} P) = S \otimes_{\mathbb{F}} J^n P$. \square

Remark. For $n = 0, 1$ and M the one-dimensional trivial module, Lemma 2 was already proved in [5, Satz II.3.5].

Let L be a finite-dimensional restricted Lie algebra and let $\chi \in L^*$. If B is a block of $u(L, \chi)$, then $\text{Irr}(B)$ will denote the set of isomorphism classes of simple $u(L, \chi)$ -modules which belong to B . Consider

$$G_0^L(B) := \{\gamma \in G_0^L \mid \gamma \cdot [M] \in \text{Irr}(B) \quad \forall [M] \in \text{Irr}(B)\}.$$

It is clear from [8, Lemma 1(b)] that $G_0^L(B)$ is a subgroup of G_0^L . Then the following result complements [8, Theorem 3].

Theorem 2 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an arbitrary field \mathbb{F} , let $\chi \in L^*$, and let B be a block ideal of $u(L, \chi)$. If M and N are simple $u(L, \chi)$ -modules, then the following statements are equivalent:*

- (a) M and N belong to B .
- (b) There exists $\gamma \in G_0^L(B)$ such that $N \cong F_\gamma \otimes_{\mathbb{F}} M$.
- (c) There exists $\gamma \in G_0^L(B)$ such that $P_L(N) \cong F_\gamma \otimes_{\mathbb{F}} P_L(M)$.
- (d) There exists $\gamma \in G_0^L(B)$ such that

$$[\text{Jac}(u(L, \chi))]^n P_L(N) \cong F_\gamma \otimes_{\mathbb{F}} [\text{Jac}(u(L, \chi))]^n P_L(M)$$

for every non-negative integer n .

Proof. (a) \implies (b): By virtue of [8, Theorem 3], there exists $\gamma \in G_0^L$ such that $N \cong F_\gamma \otimes_{\mathbb{F}} M$. Let X be an arbitrary simple $u(L, \chi)$ -module belonging to B . Since M belongs to B , [8, Lemma 1(b)] yields the existence of simple $u(L, \chi)$ -modules $X = X_0, \dots, X_m = M$ such that $\text{Ext}_{u(L, \chi)}^1(X_{i-1}, X_i) \neq 0$ or $\text{Ext}_{u(L, \chi)}^1(X_i, X_{i-1}) \neq 0$ for every $1 \leq i \leq m$. Hence

$$\text{Ext}_{u(L, \chi)}^1(F_\gamma \otimes_{\mathbb{F}} Y, F_\gamma \otimes_{\mathbb{F}} Z) \cong \text{Ext}_{u(L, \chi)}^1(Y, Z)$$

in conjunction with another application of [8, Lemma 1(b)] implies that $F_\gamma \otimes_{\mathbb{F}} X$ and $F_\gamma \otimes_{\mathbb{F}} M \cong N$ belong to B , i.e., $\gamma \in G_0^L(B)$.

Since (b) \implies (d) is just Lemma 2 and (d) \implies (c) as well as (b) \implies (a) are trivial, it remains to show the implication (c) \implies (b) which can be read off from [10, Lemma 1]. \square

Remark. If one defines $G^L(B)$ as the subgroup of G^L which permutes the simple modules in the block B , then in [4, Lemma 4.2] it is shown by using primitive central idempotents that $G^L(B)$ acts transitively on $\text{Irr}(B)$. But because $T_p(L)$ acts on every simple module belonging to B via the same *central character* $\zeta = \zeta_B$ (or since as a consequence of Theorem 2 the simple modules belonging to B have the *same* eigenvalue function), we obtain that $\gamma|_{T_p(L)} = 0$ for every $\gamma \in G^L(B)$. Hence $\gamma \in G_0^L$, i.e., $G^L(B) = G_0^L(B)$, and therefore [4, Lemma 4.2] also follows from Theorem 2.

It is an immediate consequence of [8, Corollary 2] and Theorem 2 that several invariants of the projective indecomposable $u(L, \chi)$ -modules belonging to same block coincide.

Corollary 2 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an arbitrary field \mathbb{F} and let $\chi \in L^*$. If M and N are simple $u(L, \chi)$ -modules which belong to the same block, then $P_L(M)$ and $P_L(N)$ have the same Loewy length (composition length, dimension, respectively). \square*

Remark. As in the proof of [7, Corollary 2] it follows from Corollary 1 that the composition length of every projective indecomposable $u(L, \chi)$ -module for a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} is a p -power.

We also obtain from Theorem 2 the following improved version of [8, Corollary 1 and Corollary 5] which gives a partial answer to [8, Question 2].

Corollary 3 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an arbitrary field \mathbb{F} and let $\chi \in L^*$. If B is a block of $u(L, \chi)$, then the following statements hold:*

- (a) $|\text{Irr}(B)|$ divides $|\text{Irr}(B_0(L))|$. In particular, $|\text{Irr}(B)|$ is a p -power.
- (b) If \mathbb{F} is algebraically closed, then there exists a χ -admissible polarization K such that $|\text{Irr}(B)|$ divides $p^{\dim_{\mathbb{F}} T_{\max}(K)/T_p(L)}$ for any maximal torus $T_{\max}(K)$ of K .

Proof. If S is a simple module belonging to B , then consider the stabilizer

$$\text{Stab}_{G_0^L(B)}(S) := \{\gamma \in G_0^L(B) \mid \gamma \cdot [S] = [S]\}$$

of S in $G_0^L(B)$. Since the implication (a) \implies (b) of Theorem 2 just means that $G_0^L(B)$ acts *transitively* on $\text{Irr}(B)$, we have

$$|\text{Irr}(B)| = \frac{|G_0^L(B)|}{|\text{Stab}_{G_0^L(B)}(S)|}.$$

Hence (a) follows from the fact that $G_0^L(B)$ is a subgroup of the (finite) p -group G_0^L which by [8, Theorem 5] is isomorphic to $\text{Irr}(B_0(L))$ (see also [9, p. 96]). Finally, (b) is an immediate consequence of

$$\text{Stab}_{G_0^L(B)}(S) = G_0^L(S)$$

and the proof of [10, Proposition 2]. \square

Remark. Note that the definition of the stabilizer is *independent* of the choice of S because $G_0^L(B)$ acts *transitively* on $\text{Irr}(B)$ and G_0^L is commutative. Moreover, from the argument in the remark after Theorem 2 one can also conclude that $G^L(S) = G_0^L(S)$.

The next result generalizes [7, Theorem 2] from *nilpotent* to *supersolvable* restricted Lie algebras. In the special case of *strongly solvable* restricted Lie algebras the first statement was already proved by J. C. Jantzen (see [13, 8.11(3)]).

Theorem 3 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} and let $\chi \in L^*$. If S is a simple $u(L, \chi)$ -module, then*

$$S^* \otimes_{\mathbb{F}} P_L(S) \cong \bigoplus_{\gamma \in G_0^L(S)} P_L(F_\gamma).$$

In particular, $S^ \otimes_{\mathbb{F}} P_L(S)$ belongs to the principal block of L .*

Proof. Consider $P := S^* \otimes_{\mathbb{F}} P_L(S)$ which in view of [6, Lemma 2.3] is a projective $u(L, 0)$ -module. Since $\{P_L(X) \mid X \in \text{Irr}(L, 0)\}$ is a full set of representatives of isomorphism classes of projective indecomposable $u(L, 0)$ -modules, there exist non-negative integers m_X such that

$$P \cong \bigoplus_{X \in \text{Irr}(L, 0)} P_L(X)^{\oplus m_X}.$$

Observe that $T_p(L) \subseteq C(u(L, \chi))$ acts via the same *central character* ζ on both S and $P_L(S)$ (cf. the proof of [8, Proposition 3]). Hence

$$t \cdot (\varphi \otimes w) = (t \cdot \varphi) \otimes w + \varphi \otimes (t \cdot w) = (-\zeta(t) \cdot \varphi) \otimes w + \varphi \otimes (\zeta(t) \cdot w) = 0$$

for every $t \in T_p(L)$, $\varphi \in S^*$, $w \in P_L(S)$, i.e.,

$$T_p(L) \subseteq \text{Ann}_L(P) \subseteq \text{Ann}_L(P_L(X)) \subseteq \text{Ann}_L(X)$$

for every $X \in \text{Irr}(L, 0)$ with $m_X \neq 0$. Then [8, Theorem 5] implies that X belongs to the principal block of L . In particular, we obtain from [8, Theorem 1] that $\dim_{\mathbb{F}} X = 1$, i.e., $X = F_\gamma$ for some $\gamma \in G_0^L$.

Finally, a twofold application of [1, Lemma 1.7.5] in conjunction with the adjointness of Hom and \otimes yields

$$\begin{aligned} m_\gamma &= \dim_{\mathbb{F}} \text{Hom}_L(P, F_\gamma) \\ &= \dim_{\mathbb{F}} \text{Hom}_L(P_L(S), F_\gamma \otimes_{\mathbb{F}} S) \\ &= \begin{cases} 1 & \text{if } F_\gamma \otimes_{\mathbb{F}} S \cong S \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and especially $m_\gamma \neq 0$ if and only if $\gamma \in G_0^L(S)$. \square

Remark. According to a classical result of Zassenhaus (cf. [15, Corollary 1.4.4]), $T_p(L)$ acts on $P_L(S)$ via the eigenvalue function of S which therefore can be used in the proof of Theorem 3 instead of the central character ζ .

As an immediate consequence of Theorem 3 and Corollary 1 we can determine the order of the stabilizers.

Corollary 4 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} , let $\chi \in L^*$, and let S be a simple $u(L, \chi)$ -module. Then*

$$|G_0^L(S)| = p^{\dim_{\mathbb{F}} T_{\max}(L)/T_{\max}(K)},$$

where $T_{\max}(K)$ is a maximal torus of a χ -admissible polarization K of S and $T_{\max}(L)$ is a maximal torus of L such that $T_{\max}(L) \supseteq T_{\max}(K)$. \square

Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} and let B be an arbitrary block of L . Then Corollary 3 still leaves open whether there is a formula for $|\text{Irr}(B)|$? According to Corollary 4 and the argument in the proof of Corollary 3, this is equivalent to find a formula for $G_0^L(B)$ (see also [8, Question 1]).

In view of [7, Theorem 1(a)], one could also ask whether there is a formula for $\dim_{\mathbb{F}} B$? Because of

$$B \cong \bigoplus_{S \in \text{Irr}(B)} P_L(S)^{\oplus \dim_{\mathbb{F}} S}$$

and Theorem 2 we have

$$\dim_{\mathbb{F}} B = |\text{Irr}(B)| \cdot (\dim_{\mathbb{F}} S) \cdot (\dim_{\mathbb{F}} P_L(S))$$

for every simple module S belonging to B . But then Corollary 1 implies that

$$\dim_{\mathbb{F}} B = |\text{Irr}(B)| \cdot p^{\dim_{\mathbb{F}} L/T_{\max}(K)}$$

for an arbitrary χ -admissible polarization K of S with a maximal torus $T_{\max}(K)$ or equivalently

$$\dim_{\mathbb{F}} B = |G_0^L(B)| \cdot p^{\dim_{\mathbb{F}} L/T_{\max}(L)},$$

i.e., $\dim_{\mathbb{F}} B$ can be determined if and only if $|G_0^L(B)|$ can be determined.

In particular, we obtain from Corollary 3(a) and [8, Theorem 6] the following upper bound for the dimension of an arbitrary block ideal:

Proposition 2 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} and let $\chi \in L^*$. If B is a block ideal of $u(L, \chi)$, then $\dim_{\mathbb{F}} B$ divides $p^{\dim_{\mathbb{F}} L/T_p(L)}$. \square*

Finally, one could ask whether there is a formula for $\dim_{\mathbb{F}} \text{Jac}(B)$? Since

$$\dim_{\mathbb{F}} B/\text{Jac}(B) = |\text{Irr}(B)| \cdot (\dim_{\mathbb{F}} S)^2$$

for any simple module S belonging to B , $\dim_{\mathbb{F}} \text{Jac}(B)$ can also be determined if and only if $|G_0^L(B)|$ can be determined. Set

$$L^\lambda := \{x \in L \mid \lambda([x, \ell]) = 0 \quad \forall \ell \in L\}$$

for $\lambda \in L^*$. Then we obtain the following generalization of [7, Corollary 3] from the *nilpotent* to the *supersolvable* case.

Proposition 3 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} and let $\chi \in L^*$. If B is a block ideal of $u(L, \chi)$, then there exists a χ -admissible polarization K with linear form $\lambda \in L^*$ for a simple module in B such that*

$$\dim_{\mathbb{F}} \text{Jac}(B) = |\text{Irr}(B)| \cdot p^{\dim_{\mathbb{F}} L/L^\lambda} \cdot [p^{\dim_{\mathbb{F}} L^\lambda/T_{\max}(K)} - 1]$$

for any maximal torus $T_{\max}(K)$ of K .

Proof. Let S be an arbitrary simple module in B and let K be a χ -admissible polarization for S with linear form λ . Since K is maximal isotropic with respect to λ , we have

$$\dim_{\mathbb{F}} K = \frac{1}{2} \cdot (\dim_{\mathbb{F}} L + \dim_{\mathbb{F}} L^\lambda)$$

which in view of

$$S \cong \text{Ind}_K^L(F\lambda, \chi)$$

implies that

$$\dim_{\mathbb{F}} S = p^{\frac{1}{2} \cdot (\dim_{\mathbb{F}} L/L^\lambda)}.$$

As in the proof of [7, Corollary 3] we obtain from Corollary 1 in conjunction with the above dimension formulas for B and S that

$$\begin{aligned} \dim_{\mathbb{F}} \text{Jac}(B) &= \dim_{\mathbb{F}} B - |\text{Irr}(B)| \cdot (\dim_{\mathbb{F}} S)^2 \\ &= |\text{Irr}(B)| \cdot [(\dim_{\mathbb{F}} S) \cdot (\dim_{\mathbb{F}} P_L(S)) - (\dim_{\mathbb{F}} S)^2] \\ &= |\text{Irr}(B)| \cdot [p^{\dim_{\mathbb{F}} L/T_{\max}(K)} - p^{\dim_{\mathbb{F}} L/L^\lambda}] \\ &= |\text{Irr}(B)| \cdot p^{\dim_{\mathbb{F}} L/L^\lambda} \cdot [p^{\dim_{\mathbb{F}} L^\lambda/T_{\max}(K)} - 1] \quad \square \end{aligned}$$

In particular, we can compute from [8, Theorem 6] the dimension of the Jacobson radical of the principal block:

Corollary 5 *If L is a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} , then*

$$\dim_{\mathbb{F}} \text{Jac}(B_0(L)) = p^{\dim_{\mathbb{F}} T_{\max}(L)/T_p(L)} \cdot [p^{\dim_{\mathbb{F}} L/T_{\max}(L)} - 1]$$

for every maximal torus $T_{\max}(L)$ of L . \square

Remark. Corollary 5 can immediately be generalized to blocks with a (and thus only) one-dimensional simple module(s).

Moreover, Proposition 3 can be used to derive a *semisimplicity criterion* for an arbitrary block of $u(L, \chi)$ which generalizes [7, Remark after Corollary 3] from the *nilpotent* to the *supersolvable* case.

Corollary 6 *Let L be a finite-dimensional supersolvable restricted Lie algebra over an algebraically closed field \mathbb{F} and let $\chi \in L^*$. A block B of $u(L, \chi)$ is semisimple if and only if there exists a χ -admissible polarization K with linear form $\lambda \in L^*$ for a simple module in B such that $\dim_{\mathbb{F}} L^\lambda = \dim_{\mathbb{F}} T_{\max}(K)$ for any maximal torus $T_{\max}(K)$ of K . \square*

A block B of a finite-dimensional associative \mathbb{F} -algebra A is called *stable* if

$$\frac{\dim_{\mathbb{F}} B}{|\text{Irr}(B)|} = \frac{\dim_{\mathbb{F}} A}{|\text{Irr}(A)|},$$

and A is said to be *blockstable* if every block of A is stable (see [9]).

Since in the nilpotent case reduced universal enveloping algebras are blockstable (see [7, 9]), it is natural to ask when a block of L is stable resp. when $u(L, \chi)$ is blockstable? In the light of the above observations, we see that a block B of $u(L, \chi)$ is stable if and only if

$$|\text{Irr}(L, \chi)| = p^{\dim_{\mathbb{F}} T_{\max}(K)}$$

for any maximal torus $T_{\max}(K)$ of a χ -admissible polarization K of a simple module belonging to B .

In particular, the principal block of a finite-dimensional supersolvable restricted Lie algebra L is stable if and only if $|\text{Irr}(L, 0)| = p^{\dim_{\mathbb{F}} T_{\max}(L)}$, i.e., the upper bound of [10, Theorem 4] is realized. Moreover, we obtain from [10, Theorem 5]:

Proposition 4 *If L is a finite-dimensional strongly solvable restricted Lie algebra over an algebraically closed field \mathbb{F} , then $u(L, \chi)$ is blockstable for every $\chi \in L^*$. \square*

Remark. The example at the end of [10] shows that there exist finite-dimensional supersolvable restricted Lie algebras with *no* stable block at all.

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